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VIBRATIONS OF THICK AND THIN CYLINDRICAL SHELLS SURROUNDED BY WATER

By

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Errata Sheet No. 1

"Vibrations of Thick and Thin Cylindrical Shells Surrounded by Water," J. Greenspon, Nonr - 2733(00) Tech. Rep. No. 4

In the Radially Pulsating Cylinder (Part IV E) the expression for the loaded Q is incorrect in the report; it should be as follows:

$$Q = \frac{1 + \frac{h^2}{12a^2}}{(\Omega)_{air}(\Omega)_{water}\frac{\delta}{\pi} + (\Omega)_{water}\frac{\rho_0}{\rho_t}\frac{c_0}{c_p}\frac{a}{h}\theta_{00}}$$

Thus for the thin cylinder $(\Omega)_{air} \approx 1$

$$\frac{1}{Q} \approx (\Omega)_{water}\frac{\delta}{\pi} + (\Omega)_{water}\frac{\rho_0}{\rho_t}\frac{c_0}{c_p}\frac{a}{h}\theta_{00}$$

The efficiency of the transducer is as follows:

$$\text{Efficiency} \approx \frac{\frac{\rho_0}{\rho_t}\frac{c_0}{c_p}\frac{a}{h}\theta_{00}}{\frac{\delta}{\pi} + \frac{\rho_0}{\rho_t}\frac{c_0}{c_p}\frac{a}{h}\theta_{00}}$$

The Q and efficiency of the steel radiator with $\delta = 0.02$, $\rho_0/\rho_t = 0.127$, $c_0/c_p = 0.278$, $a/h = 20$, $\theta_{00} = 1$ which resonated in water at $\Omega_w = 1$ is now as follows:

$$Q = \frac{1}{\frac{0.02}{3.14} + 0.127 \times 0.278 \times 20 \times 1} = 1.4$$

Efficiency = 99 %

The Q and efficiency of the plexiglass radiator with $\delta = 2$, $\rho_0/\rho_t = 0.85$, $c_0/c_p = 0.93$, $a/h = 10$, $\theta_{00} = 0.32$ which resonated at $\Omega_w \approx 0.2$ is now as follows:

$$Q = \frac{1}{\frac{0.2 \times 2}{3.14} + 0.2 \times 0.85 \times 0.93 \times 10 \times 0.32} = 1.6$$

Efficiency = 80 %

VIBRATIONS OF THICK AND THIN CYLINDRICAL SHELLS SURROUNDED BY WATER

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LIST OF SYMBOLS

u_r	radial displacement in a thick shell
u_θ	tangential displacement in a thick shell
u_z	longitudinal displacement in a thick shell
r	radial direction
θ	tangential direction
z	longitudinal direction
m	number of axial half waves in the vibration pattern on a finite cylinder
ℓ	length of finite cylinder
n	number of circumferential waves in vibration pattern of cylinder
p	natural frequency
$rr, r\theta, rz$	stresses on cylindrical surface of a thick cylinder
$R_1(r), R_2(r), R_3(r)$	functions describing the distribution of the radial, tangential and longitudinal displacements respectively, as a function of the radial coordinate
p_{f_o}	acoustic pressure in the surrounding fluid medium
ω	forcing frequency of harmonic forces applied to the shell
$F_{mn}(r)$	function describing the distribution of pressure in the surrounding medium as a function of the radial coordinate
λ_m	longitudinal wave length for the mth mode (m=1,2,3...)
p_i	internal forcing pressure
P_{mn}	Fourier component of internal forcing pressure
ρ_o	density of surrounding fluid
c_o	sound velocity in surrounding fluid
$\alpha_{mn}(a_o)$	amplitude of the radial displacement of the mnth mode evaluated at the outside surface of the cylinder
a_i	inside radius of cylinder
a_o	outside radius of cylinder
$\zeta_{mn}(k_m a_o) = \theta_{mn}(k_m a_o) + i\,\chi_{mn}(k_m a_o)$	the acoustic impedance
θ_{mn}	resistive impedance

X_{mn} reactive impedance

$k_m = [\omega^2/c_o^2 - (2\pi/\lambda_m)^2]^{1/2}$

$(U_r)_{mn}$ radial displacement of mnth mode of a thick shell

$(U_\theta)_{mn}$ tangential displacement of the mnth mode of a thick shell

$(U_z)_{mn}$ longitudinal displacement of the mnth mode of a thick shell

$C_1 \ldots C_6$ constants of integration for thick shell solution

$C_1' \ldots C_6'$ real part of $C_1 \ldots C_6$ respectively

$C_1'' \ldots C_6''$ imaginary part of $C_1 \ldots C_6$ respectively

A', B', C', D', E', F' deflection constants for thick shell solution

$a_1 \ldots a_6, b_1 \ldots b_6$ coupling constants between deflection and pressure

I_n, K_n Bessel Functions of imaginary argument

J_n, Y_n Bessel Functions of real argument

μ shear modulus for cylinder material

E modulus of elasticity for cylinder material

C_d velocity of an elastic dilatational wave

C_r velocity of an elastic rotational wave

ρ_t density of the cylinder material

ν Poisson's ratio for cylinder

$\bar{K} = \frac{1}{2} \beta \psi \, c_o/c_r \, \rho_o/\rho_t$

$\beta = \pi d_o/\lambda_m$ $\beta = m\pi a_o/\ell$ (for finite cylinder)

d_o outside diameter of cylinder

$\psi = \dfrac{\omega}{2\pi/\lambda_m [\mu/\rho]^{1/2}}$ forcing frequency parameter (resonance occurs at $\omega = p$ or $\psi = \psi_r$

$\psi_r = c/c_r$ ratio of phase velocity of waves to velocity of rotational wave

$a_{11} \ldots a_{66}$ determinant constants

P_o amplitude of forcing pressure

$f(\theta, z)$ distribution of forcing pressure

R nondimensional quantity proportional to radial deflection at outside surface of cylinder (exact theory)

P nondimensional quantity proportional to fluid pressure at at outside surface of cylinder (exact theory)

$\alpha = a_i/a_o$

ω forcing frequency

$(p_{fo})_{mn}$ pressure in fluid for mnth mode

ρ_i density of fluid inside tube

C_i sound velocity of fluid inside tube

$C_p = \sqrt{E/\rho_t(1-\nu^2)}$

p_{f_i} inside fluid pressure due to vibration of tube

p_{s_o} outside static pressure in medium

p_{s_i} inside static pressure in tube

h thickness of tube $=(a_o - a_i)$

a mean radius of tube $= (a_o + a_i)/2 = \left(\frac{1+\alpha}{2}\right) a_o$

U_{mn}, V_{mn}, W_{mn} longitudinal, tangential and radial displacements of the midsurface of the tube (thin shell)

A_{mn}, B_{mn}, C_{mn} amplitudes of the longitudinal, tangential and radial displacements (thin shell)

$\bar{\lambda} = \pi d/\lambda_m = m\pi a/l = \frac{1+\alpha}{2}\beta$

d mean diameter

$k_m a = \sqrt{(\bar{\lambda}\psi^{c}/c_o)^2 - \bar{\lambda}^2}$

$k_m' a = \sqrt{\bar{\lambda}^2 - (\lambda\psi^{c}/c_o)^2}$

$p_{mn}(r)$ radial distribution of internal fluid pressure

$a_{11}' \cdots a_{33}', b_{11}' \cdots b_{33}'$ determinant constants (thin shell)

S damping constant

$\bar{\alpha} = h/a$ (thin shell)

$p = \frac{\Omega}{a_o}\sqrt{E/\rho_t(1-\nu^2)}$; $\Omega = \psi\beta\sqrt{\frac{1-\nu}{2}}$ (for thick shell theory)

$p = \frac{\bar{\Omega}}{a}\sqrt{E/\rho_t(1-\nu^2)}$; $\bar{\Omega} = \psi\bar{\lambda}\sqrt{\frac{1-\nu}{2}}$ (for thin shell theory)

γ_{mn}, γ' parameters associated with internal fluid impedance

θ' parameter associated with external fluid resistance

χ' parameter associated with external fluid reactance

$\bar{n}/p$ ratio of damping to critical damping

A_r, B_r, C_r real part of Amn, Bmn, Cmn respectively

A_i, B_i, C_i imaginary part of Amn, Bmn, Cmn respectively

σ_θ tangential stress in thin shell theory

σ_z	longitudinal stress in thin shell theory
ϵ_θ, ϵ_z	strains in thin shell theory
$\bar{u}$, $\bar{v}$, $\bar{w}$	nondimensional deflection parameters
$\bar{p}_o$, $\bar{p}$	nondimensional pressure parameters
$\bar{\sigma}_\theta$, $\bar{\sigma}_z$	nondimensional stress parameters
φ_{mn}	phase angle

ABSTRACT

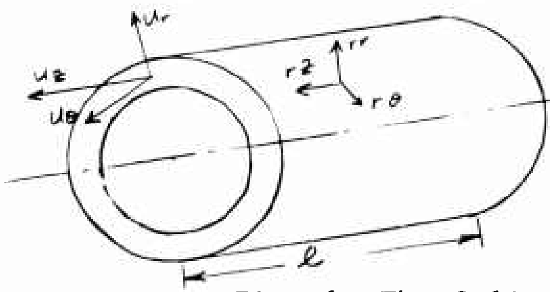
This report treats the free and forced vibrations of infinitely long pressureized cylindrical shells surrounded by water and containing fluid. Exact elasticity theory is used to treat unpressureized shells and an approximate shell theory is employed to treat the effects of static pressure, internal fluid, and structural damping. A study is made of the effects of these parameters on the dynamic behavior of the shell. Comparisons are made between the results of the exact and approximate theories.

I. INTRODUCTION

There have been a number of previous studies on the vibration of infinitely long thin cylindrical shells in water.[1-5] Although the infinite shell solution cannot be expected to describe the complete behavior of a finite shell accurately, it can be used to point out a number of important characteristics such as the approximate frequency-wave length spectrum, the reduction of the natural frequency due to presence of the water, and the approximate magnitude and directivity of the sound field. Approximate numerical solutions will eventually have to be used to obtain an accurate solution for a finite shell vibrating in water, but an overall picture of the behavior can undoubtedly be obtained by studying the results based on the infinite shell solution.

It should be made clear at the onset however, just how the infinite shell solution is to be used and what characteristics it can be expected to describe for a finite shell.

If we consider a finite thick shell with freely supported ends vibrating in a vacuum,[6] the displacement pattern for standing vibrations can be represented as follows:



$$
\begin{aligned}
(U_r)_{mn} &= R_1(r)\sin\frac{m\pi z}{l}\cos n\theta\, e^{i\omega t}\\
(U_\theta)_{mn} &= R_2(r)\sin\frac{m\pi z}{l}\sin n\theta\, e^{i\omega t}\\
(U_z)_{mn} &= R_3(r)\cos\frac{m\pi z}{l}\cos n\theta\, e^{i\omega t}
\end{aligned}
\qquad [1]
$$

Fig. 1. The Cylinder

1. M. C. Junger, J. Acoust. Soc. Am., 25, 40-47 (1953).
2. H. H. Bleich and M. L. Baron, Jour. Appl. Mech., June, 1954.
3. H. H. Bleich, Proc. of 2nd U. S. Nat. Cong. Appl. Mech. (1954).
4. M. C. Junger, Jour. Appl. Mech., 74, 439-445, (1952).
5. Z.V. Kolotikhina, Soviet Physics - Acoustics, 4, 4, 344-351 (1958).
6. J. E. Greenspon, "Vibrations of Thick Shells in a Vacuum," Office of Naval Research, Project No. NR 385-412, Contract No. Nonr - 2733(00), Tech. Rep. No. 1, Feb., 1959.

It has been shown by Arnold and Warburton[7,8] that such displacement functions represent realistic end conditions and can even be used to approximate fixed ends if the longitudinal wave length parameter is redefined.

If it is assumed that the shell is extended to infinity in both directions along the axis, the displacement pattern will be the same as above with the wave length of the motion being $\lambda_m = 2l/m$. In other words, the vibration pattern on a finite freely supported cylinder of length l is represented by two sinusoidal waves traveling in opposite directions on the infinitely long cylinder.

Now if the infinitely long cylinder is placed in the water the pressure produced in the water due to the vibration of the cylinder is as follows:[1]

$$p_{fo}(r,\theta,z) = e^{i\omega t}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} F_{mn}(r)\cos n\theta \sin\frac{2\pi z}{\lambda_m} \qquad [2]$$

and for each displacement pattern

$$(u_r)_{mn} = R_1(r)\sin\frac{2\pi z}{\lambda_m}\cos n\theta \, e^{i\omega t} \qquad [3]$$

there is a pressure pattern,

$$(p_{fo})_{mn} = F_{mn}(r)\cos n\theta \sin\frac{2\pi z}{\lambda_m} e^{i\omega t} \qquad [4]$$

Thus each elastic mode of given m and n excites a single pressure mode in the fluid.

In the finite shell there will be no direct coupling such as this because of the presence of the ends of the shell. We are therefore making the following assumptions in applying the infinite shell solution to the finite shell with freely supported ends vibrating in water:

1. The pressure produced on the portion of the infinite shell from A to B (see Fig. 2) by the adjacent portions (∞ to A and B to ∞) is small

Fig. 2 Wave Pattern on the Infinite Cylinder

2. The motion of the ends of the actual finite shell do not effect the pressure on the cylindrical surface.

---

7. R. N. Arnold and G. B. Warburton, Proc. Roy. Soc., 197, Series A, 238-256 (1949).
8. R. N. Arnold and G. B. Warburton, Proc. of the Institution of Mech. Engrs., 167, 62-74 (1953).

For modes in which the longitudinal displacement is small compared to the radial and tangential displacements, assumption 2 should be valid. Assumption 1 is incorrect for a finite shell but it is believed that such characteristics as the frequency - wave length spectrum and the relative pressures excited by different modes of the shell can be obtained satisfactorily. The vibrating portions of the infinite shell which are far away from A and B will have a very small effect on the part between A and B. However the parts near A and B will have an appreciable effect.

A previous reference[9] presents the characteristics of the axially symmetric modes (n=0) of infinitely long thick cylindrical shells vibrating in water. Very special types of loading have to be applied to the cylinder to excite these modes and their natural frequencies are usually high. In spite of these facts, these modes have been the most useful ones for transducer applications because of the relatively large sound power radiated for a given deflection.

In this paper the more general type of deformation corresponding to $n \geq 1$ will be considered in addition to n=0. The modes in which n=1 and $n \geq 2$ are known as the beam mode and lobar modes respectively, and are usually the ones excited by general types of transverse loads applied to the surface of the shell. Modes of this type have been known to produce unwanted noise radiation in submarine hulls. It is possible that their low frequency characteristics combined with their directivity possibilities could prove useful in transducer applications.

II. THICK SHELL THEORY WITHOUT INTERNAL FLUID OR PRESSURE EFFECTS

The theory of nonaxially symmetric vibrations of thick cylindrical shells in an acoustic medium follows the axially symmetric theory[9] rather closely with the exception that two more boundary conditions must be satisfied on the cylindrical surfaces for the nonaxially symmetric (flexural) case and the displacements are now dependent on θ.

For these nonaxially symmetric vibrations the boundary conditions to be satisfied on the cylindrical interface between the fluid and shell and on the inside shell surface are given in Eq. [5] (see Fig. 1 for notation).

9. J. E. Greenspon, J. Acoust. Soc. Am., 32, 1017-1025 (1960).

$$
\begin{aligned}
rr(a_o, \theta, z, t) &= p_{f_o}(a_o, z, t) \\
rr(a_i, \theta, z, t) &= p_i(\theta, z, t) \\
r\theta(a_o, \theta, z, t) &= 0 \\
r\theta(a_i, \theta, z, t) &= 0 \\
rz(a_o, \theta, z, t) &= 0 \\
rz(a_i, \theta, z, t) &= 0
\end{aligned} \qquad [5]
$$

The first boundary condition states that the normal stress on the outside cylindrical surface of the shell is equal to the pressure in the fluid at this surface. The second equation states that the normal stress on the inside cylindrical surface is equal to the pressure applied by external means to the inside surface; this pressure will be assumed harmonic in time. The remaining four equations state that there are no shear stresses acting on the shell surfaces, the fluid being assumed non-viscous.

It will further be assumed that the internal pressure is such that it can be expanded into a Fourier Series as follows:

$$p_i(\theta, z, t) = e^{i\omega t} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} p_{mn} \sin \frac{2\pi z}{\lambda_m} \cos n\theta \qquad [6]$$

It has been shown[1] that the outside fluid pressure can be expanded into a similar Fourier Series for the infinite cylinder. By substitution of the expressions for the internal pressure and the outside fluid pressure into [5], the following equations are obtained:

$$
\begin{aligned}
rr(a_o) &= i\omega \rho_o c_o \alpha_{mn}(a_o) \mathcal{J}_{mn}(k_m a_o) \\
rr(a_i) &= p_{mn} \\
r\theta(a_o) &= r\theta(a_i) = rz(a_o) = rz(a_i) = 0
\end{aligned} \qquad [7]
$$

where

ω = forcing frqquency of internal pressure
ρ_o = density of the fluid surrounding the shell
c_o = sound velocity in the surrounding fluid
$\alpha_{mn}(a_o)$ = amplitude of the radial displacement of the mnth mode evaluated at the outside cylindrical surface

$\mathcal{J}_{mn} = \theta_{mn} + i\chi_{mn}$ the acoustic impedance
$k_m = [\omega^2/c_o^2 - (2\pi/\lambda_m)^2]^{1/2}$
λ_m = wave length of the vibration in the longitudinal direction

It has been shown previously[6] that the displacements u_r, u_θ, u_z can be written in terms of six arbitrary constatnts $C_1 \ldots\ldots C_6$. Therefore the radial deflection u_r and the fluid pressure p_{f_0} at the outside interface between the cylinder and the surrounding fluid for the mnth mode can be written as follows:

$$(u_r)_{mn} = \frac{1}{a_0}\Big[(C_1'A' + C_2'B' + C_3'C' + C_4'D' + C_5'E' + C_6'F')^2 + (C_1''A' + C_2''B' + C_3''C' + C_4''D' + C_5''E' + C_6''F')^2\Big]^{1/2} \cos n\theta \sin\frac{2\pi z}{\lambda_m}\, e^{i(\omega t - \phi_{mn})} \qquad [8]$$

$$(p_{f_0})_{mn} = \frac{2\mu}{a_0^2}\Big\{\Big[(a_1C_1' - b_1C_1'') + (a_2C_2' - b_2C_2'') + (a_3C_3' - b_3C_3'') + (a_4C_4' - b_4C_4'') + (a_5C_5' - b_5C_5'') + (a_6C_6' - b_6C_6'')\Big]^2 + \Big[(b_1C_1' + a_1C_1'') + (b_2C_2' + a_2C_2'') + (b_3C_3' + a_3C_3'') + (b_4C_4' + a_4C_4'') + (b_5C_5' + a_5C_5'') + (b_6C_6' + a_6C_6'')\Big]\Big\}^{1/2} \cos n\theta \sin\frac{2\pi z}{\lambda_m}\, e^{i(\omega t - \phi_{mn})} \qquad [9]$$

where $A' \ldots\ldots F'$, $a_1 \ldots a_6$, $b_1 \ldots b_6$ are as follows:

Table 1. Deflection Constants

	If $\omega/c_d < 2\pi/\lambda_m$, $\omega/c_r < 2\pi/\lambda_m$	If $\omega/c_d < 2\pi/\lambda_m$, $\omega/c_r > 2\pi/\lambda_m$	If $\omega/c_d > 2\pi/\lambda_m$, $\omega/c_r > 2\pi/\lambda_m$
A'	$\eta I_{n-1}(\eta) - n I_n(\eta)$	$\eta I_{n-1}(\eta) - n I_n(\eta)$	$\eta J_{n-1}(\eta) - n J_n(\eta)$
B'	$-\eta K_{n-1}(\eta) - n K_n(\eta)$	$\eta K_{n-1}(\eta) - n K_n(\eta)$	$\eta Y_{n-1}(\eta) - n Y_n(\eta)$
C'	$\zeta I_{n-1}(\zeta) - n I_n(\zeta)$	$\zeta J_{n-1}(\zeta) - n J_n(\zeta)$	$\zeta J_{n-1}(\zeta) - n J_n(\zeta)$
D'	$-\zeta K_{n-1}(\zeta) - n K_n(\zeta)$	$\zeta Y_{n-1}(\zeta) - n Y_n(\zeta)$	$\zeta Y_{n-1}(\zeta) - n Y_n(\zeta)$
E'	$I_n(\zeta)$	$J_n(\zeta)$	$J_n(\zeta)$
F'	$K_n(\zeta)$	$Y_n(\zeta)$	$Y_n(\zeta)$

$$\bar{K} = \tfrac{1}{2}\beta\psi \frac{c_0}{c_r}\frac{\rho_0}{\rho}$$

$$\begin{array}{llll} a_1 = A'\chi_{mn}\bar{K} & a_4 = D'\chi_{mn}\bar{K} & b_1 = -A'\theta_{mn}\bar{K} & b_4 = -D'\theta_{mn}\bar{K} \\ a_2 = B'\chi_{mn}\bar{K} & a_5 = E'\chi_{mn}\bar{K} & b_2 = -B'\theta_{mn}\bar{K} & b_5 = -E'\theta_{mn}\bar{K} \\ a_3 = C'\chi_{mn}\bar{K} & a_6 = F'\chi_{mn}\bar{K} & b_3 = -C'\theta_{mn}\bar{K} & b_6 = -F'\theta_{mn}\bar{K} \end{array} \qquad [10]$$

The parameters η, β, ζ are explained completely in a previous reference[6] and also briefly in the Appendix. The quantities χ_{mn} and θ_{mn} are contained in a previous reference[1] and for completeness are given in the Appendix.

Going back to the boundary conditions (eq. [7]) and substituting the expressions for the stresses and impedance, we obtain six complex algebraic equations in the six unknown complex constants $C_1 \ldots C_6$. These equations are as follows:

$$
\begin{aligned}
&[a_{11} - (a_1 + ib_1)]C_1 + [a_{12} - (a_2 + ib_2)]C_2 + [a_{13} - (a_3 + ib_3)]C_3 \\
&+ [a_{14} - (a_4 + ib_4)]C_4 + [a_{15} - (a_5 + ib_5)]C_5 + [a_{16} - (a_6 + ib_6)]C_6 = 0 \\
&a_{21} C_1 + a_{22} C_2 + a_{23} C_3 + a_{24} C_4 + a_{25} C_5 + a_{26} C_6 = \frac{P_{mn} a_0^2}{2\mu} \\
&a_{31} C_1 + a_{32} C_2 + a_{33} C_3 + a_{34} C_4 + a_{35} C_5 + a_{36} C_6 = 0 \\
&a_{41} C_1 + a_{42} C_2 + a_{43} C_3 + a_{44} C_4 + a_{45} C_5 + a_{46} C_6 = 0 \\
&a_{51} C_1 + a_{52} C_2 + a_{53} C_3 + a_{54} C_4 + a_{55} C_5 + a_{56} C_6 = 0 \\
&a_{61} C_1 + a_{62} C_2 + a_{63} C_3 + a_{64} C_4 + a_{65} C_5 + a_{66} C_6 = 0
\end{aligned}
\qquad [11]
$$

where $C_1 = C_1' + iC_1''$, $C_2 = C_2' + iC_2''$, $C_3 = C_3' + iC_3''$

$C_4 = C_4' + iC_4''$, $C_5 = C_5' + iC_5''$, $C_6 = C_6' + iC_6''$

and where $a_{11} \ldots a_{66}$ are the coefficients for flexural vibrations as contained in a previous reference[6] and also in the Appendix of this report.

Let the internal pressure be written as follows:

$$p_i(\theta, z, t) = P_0 f(\theta, z) e^{i\omega t} \qquad [12]$$

The deflection and pressure at the outside interface of the cylinder can then be written

$$(u_r)_{mn} = \left[\frac{2P_0}{\pi\lambda_m}\frac{a_0}{2\mu}\int_0^{\lambda_m}\int_0^{2\pi} f(\theta,z)\sin\frac{2\pi z}{\lambda_m}\cos n\theta \, dz\, d\theta\right] R \sin\frac{2\pi z}{\lambda_m}\cos n\theta\, e^{i(\omega t-\phi_{mn})}$$

$$(p_{f_0})_{mn} = \left[\frac{2P_0}{\pi\lambda_m}\int_0^{\lambda_m}\int_0^{2\pi} f(\theta,z)\sin\frac{2\pi z}{\lambda_m}\cos n\theta\, dz\, d\theta\right] P \sin\frac{2\pi z}{\lambda_m}\cos n\theta\, e^{i(\omega t-\phi_{mn})}$$

ϕ_{mn} is the phase angle [13]

where R and P are nondimensional quantities which have the following values:

$$R = 2\mu/P_{mn}a_0{}^2 \left[\text{Bracket in Eq. [8]}\right]$$

$$P = 2\mu/P_{mn}a_0{}^2 \left\{\text{Brace in Eq. [9]}\right\}$$ [14]

The nondimensional quantities R and P are functions of the forcing frequency ω, the thickness ratio $\alpha = a_i/a_0$, the wave length ratio $\pi d_0/\lambda_m$, the circumferential parameter n, Poisson's ratio ν, the wave velocity ratio c_0/c_r and the density ratio ρ_0/ρ_t

The expressions [14] can be interpreted as being the nondimensional transfer function due to the Fourier component pressure P_{mn}. For a finite cylinder of length ℓ, $\lambda_m = \frac{2\ell}{m}$ so $\pi d_0/\lambda_m = m\pi a_0/\ell$ (assuming for the moment that the theory was correct for a finite cylinder). Thus expressions [13] and [14] give the deflection and pressure in the mnth mode (i.e. for a given nodal pattern m and n) as a function of the frequency ω. Thus for a given cylinder of physical parameters a_i/a_0, d_0/ℓ, n, ν vibrating in fluid with parameters c_0/c_r, ρ_0/ρ_t, expressions [13] and [14] are the deflection and pressure response factor in each mode as a function of frequency. The trace of R vs ω will be analogous to a single degree of freedom (mass spring system) resonance curve which starts out at a static response and peaks at the individual frequency of each mode. The response to any load distribution can then be written as

$$u_r = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(u_r)_{mn}$$

$$p_{f_0} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(p_{f_0})_{mn}$$ [15]

If we wish to uncouple the modes completely we can apply a pressure which has the same distribution as the deflection, i. e.

$$p_i(\theta,z) = P_0 \sin 2\pi z/\lambda_m \cos n\theta\, e^{i\omega t}$$

In this case the deflection and pressure are

$$u_r = (u_r)_{mn} = \frac{P_0 a_0}{2\mu} R \sin\frac{2\pi z}{\lambda_m}\cos n\theta\, e^{i(\omega t-\phi_{mn})}$$ [16]

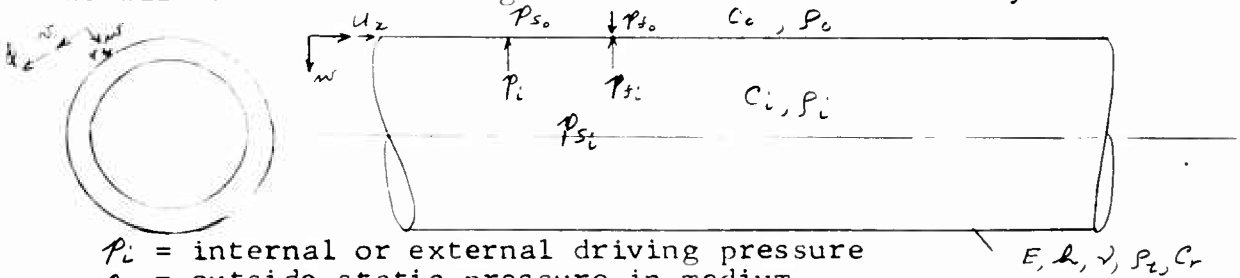
$$p_{f_0} = (p_{f_0})_{mn} = P_0 (P) \sin\frac{2\pi z}{\lambda_m}\cos n\theta\, e^{i(\omega t-\phi_{mn})}$$

The same theory also applies to elastic waves traveling along tubes which are immersed in water in the same manner as described in a previous reference[6]. For this case we can plot $\gamma = C/C_r$ vs $\beta = \pi d_o/\lambda_m$ which gives the dispersion curve for elastic waves traveling along the infinite tube.

III. THIN SHELL THEORY WITH INTERNAL FLUID AND PRESSURE

The exact theory as given in section II is quite cumbersome to work with and requires long computation times even on the electronic computer. Therefore, for practical purposes an approximate theory was developed including the additional effects of internal and external static pressure, internal fluid, and structural damping in addition to the effect of the outside acoustic medium. The comparisons between results of the approximate theory with those of the exact theory demonstrate that the approximate theory can be applied for rather thick shells.

We will use the following nomenclature for the theory:



p_i = internal or external driving pressure
p_{s_o} = outside static pressure in medium
p_{s_i} = inside static pressure
p_{f_o} = outside fluid pressure due to vibration of tube
p_{f_i} = inside fluid pressure due to vibration of tube
E = modulus of elasticity of tube material
ν = Poisson's ratio for tube material
h = thickness of tube
a = mean radius of tube
ρ_t = density of tube material
C_o = velocity of sound in outside medium
ρ_o = density of surrounding fluid
C_i = velocity of sound in tube fluid
ρ_i = density of fluid inside tube
C_r = velocity of rotational wave in tube

Fig. 3 Pressurized Cylinder with Fluid

The Flugge[10] shell equations with the addition of structural damping, inside and outside fluid are as follows:

$$
\begin{aligned}
&a^2\frac{\partial^2 u}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2 u}{\partial\phi^2}+\nu a\frac{\partial w}{\partial x}+\frac{1+\nu}{2}a\frac{\partial^2 v}{\partial x\partial\phi}+\frac{h^2}{12a^2}\left[\frac{1-\nu}{2}\frac{\partial^2 u}{\partial\phi^2}-a^3\frac{\partial^3 w}{\partial x^3}+\frac{1-\nu}{2}a\frac{\partial^3 w}{\partial x\partial\phi^2}\right]\\
&\quad-\frac{\rho_t a^2}{E}(1-\nu^2)\frac{\partial^2 u}{\partial t^2}-\frac{a^2(1-\nu^2)}{Eh}\kappa\frac{\partial u}{\partial t}=\frac{(p_{s_i}-p_{s_o})a(1-\nu^2)}{Eh}\left(a\frac{\partial w}{\partial x}-\frac{\partial^2 u}{\partial\phi^2}\right)\\
&\frac{1+\nu}{2}a\frac{\partial^2 u}{\partial x\partial\phi}+\frac{\partial^2 v}{\partial\phi^2}+\frac{1-\nu}{2}a^2\frac{\partial^2 v}{\partial x^2}+\frac{\partial w}{\partial\phi}+\frac{h^2}{12a^2}\left[\frac{3(1-\nu)}{2}a^2\frac{\partial^2 v}{\partial x^2}-\frac{3-\nu}{2}a^2\frac{\partial^3 w}{\partial x^2\partial\phi}\right]\\
&\quad-\frac{\rho_t a^2}{E}(1-\nu^2)\frac{\partial^2 v}{\partial t^2}-\frac{a^2(1-\nu^2)}{Eh}\kappa\frac{\partial v}{\partial t}=-\frac{1}{2}\frac{(p_{s_i}-p_{s_o})a(1-\nu^2)}{Eh}a^2\frac{\partial^2 v}{\partial x^2}\\
&\nu a\frac{\partial u}{\partial x}+\frac{\partial v}{\partial\phi}+w+\frac{h^2}{12a^2}\left[\frac{1-\nu}{2}a\frac{\partial^3 u}{\partial x\partial\phi^2}-a^3\frac{\partial^3 u}{\partial x^3}-\frac{3-\nu}{2}a^2\frac{\partial^3 v}{\partial x^2\partial\phi}+a^4\frac{\partial^4 w}{\partial x^4}+2a^2\frac{\partial^4 w}{\partial x^2\partial\phi^2}\right.\\
&\quad\left.+\frac{\partial^4 w}{\partial\phi^4}+2\frac{\partial^2 w}{\partial\phi^2}+w\right]+\frac{\rho_t a^2}{E}(1-\nu^2)\frac{\partial^2 w}{\partial t^2}+\frac{a^2(1-\nu^2)}{Eh}\kappa\frac{\partial w}{\partial t}=\\
&\frac{Pa^2(1-\nu^2)}{Eh}+\frac{a^2(1-\nu^2)}{Eh}\left[p_{f_i}-p_{f_o}\right]+\frac{(p_{s_i}-p_{s_o})a(1-\nu^2)}{Eh}\left[\frac{\partial^2 w}{\partial\phi^2}+a\frac{\partial u}{\partial x}+w\right]+\frac{1}{2}\frac{(p_{s_i}-p_{s_o})a(1-\nu^2)}{Eh}a^2\frac{\partial^2 w}{\partial x^2}
\end{aligned}
\qquad [17]
$$

In the above equations $\kappa\,\partial u/\partial t$, $\kappa\,\partial v/\partial t$, $\kappa\,\partial w/\partial t$ are the structural damping forces per unit area in the x, ϕ and r directions respectively.

The displacement components for the infinitely long shell are taken as follows:

$$
\begin{aligned}
u_{mn}&=A_{mn}\cos 2\pi x/\lambda_m \cos n\phi\, e^{i\omega t}\\
v_{mn}&=B_{mn}\sin 2\pi x/\lambda_m \sin n\phi\, e^{i\omega t}\\
w_{mn}&=C_{mn}\sin 2\pi x/\lambda_m \cos n\phi\, e^{i\omega t}
\end{aligned}
\qquad [18]
$$

where Amn, Bmn, Cmn are the amplitudes of the displacements in the mnth mode. The fluid pressure in the surrounding fluid can be written as follows:

$$(p_{f_o})_{mn}=i\omega\rho_o c_o C_{mn} J_{mn}(k_m r)\cos n\phi \sin\frac{2\pi x}{\lambda_m} e^{i\omega t}$$

where J_{mn} is the acoustic impedance of the fluid as described before. In thin shell theory it is assumed that loads are applied at the median surface, therefore the acoustic impedance is as follows:

$$J_{mn}=\theta_{mn}+i\chi_{mn}$$

If $\psi c_r/c_o>1$

$$\chi_{mn}(k_m a)=\frac{-\bar{\lambda}\psi c_r/c_o\left\{J_n(k_m a)\left[-J_{n+1}(k_m a)+\frac{n}{k_m a}J_n(k_m a)\right]+Y_n(k_m a)\left[-Y_{n+1}(k_m a)+\frac{n}{k_m a}Y_n(k_m a)\right]\right\}}{k_m a\left\{\left[-J_{n+1}(k_m a)+\frac{n}{k_m a}J_n(k_m a)\right]^2+\left[-Y_{n+1}(k_m a)+\frac{n}{k_m a}Y_n(k_m a)\right]^2\right\}}\qquad [19]$$

10. W. Flugge, "Statik und Dynamik der Schalen," Springer-Verlag, 1934, p. 101 and 229.

$$\theta_{mn}(k_m a) = \frac{2\bar{\lambda}\psi^{c_r/c_o}}{\pi(k_m a)^2\left\{\left[-J_{n+1}(k_m a) + \frac{n}{k_m a}J_n(k_m a)\right]^2 + \left[-Y_{n+1}(k_m a) + \frac{n}{k_m a}Y_n(k_m a)\right]^2\right\}}$$

where $k_m a = \left[(\bar{\lambda}\psi^{c_r/c_o})^2 - \bar{\lambda}^2\right]^{1/2}$

If $\psi^{c_r/c_o} < 1$

$$\chi_{mn} = -\frac{\bar{\lambda}\psi^{c_r/c_o} K_n(k_m' a)}{k_m' a\left[K_{n+1}(k_m' a) + \frac{n}{k_m' a}K_n(k_m' a)\right]} \qquad [20]$$

$$k_m' a = \left[\bar{\lambda}^2 - (\bar{\lambda}\psi^{c_r/c_o})^2\right]^{1/2}$$

The fluid pressure on the inside of the shell is the solution of the wave equation

$$c_i^2\nabla^2 p_{f_i} = \frac{\partial^2 p_{f_i}}{\partial t^2} \qquad [21]$$

For the infinitely long shell we can write

$$(p_{f_i})_{mn} = p_{mn}(r)\cos n\phi \sin\frac{2\pi x}{\lambda_m} e^{i\omega t} \qquad [22]$$

If

$$\psi^{c_r/c_i} > 1 \quad p_{mn} = D_{mn}J_n(k_r r)$$

$$k_r a = \left[(\bar{\lambda}\psi^{c_r/c_i})^2 - \bar{\lambda}^2\right]^{1/2}$$

$$\psi^{c_r/c_i} < 1 \quad p_{mn} = D_{mn}I_n(k_r' r)$$

$$k_r' a = \left[\bar{\lambda}^2 - (\bar{\lambda}\psi^{c_r/c_i})^2\right]^{1/2} \qquad [23]$$

The constant D_{mn} is determined from the boundary condition at the inner surface of the tube

$$w_{mn}(a,z) = \frac{1}{\rho_i\omega^2}\left.\frac{\partial(p_{f_i})_{mn}}{\partial r}\right]_{r=a} \qquad [24]$$

We take the internal driving pressure to be

$$p_i(\theta,z) = e^{i\omega t}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} P_{mn}\cos n\theta \sin\frac{2\pi x}{\lambda_m} \qquad [25]$$

Substituting the expressions for the desplacements and the fluid pressures into the equations of motion, the following set of equations result:

$$A_{mn}[a_{11}' + i b_{11}'] + B_{mn}[a_{12}'] + C_{mn}[a_{13}'] = 0$$

$$A_{mn}[a_{21}'] + B_{mn}[a_{22}' + i b_{22}'] + C_{mn}[a_{23}'] = 0 \qquad [26]$$

$$A_{mn}[a_{31}'] + B_{mn}[a_{32}'] + C_{mn}[a_{33}' + i b_{33}'] = \frac{a^2(1-\nu^2)}{Eh}P_{mn}$$

where

$$a_{11} = -\bar{\lambda}^2 - \frac{1-\nu}{2} n^2 - \frac{\bar{\alpha}^2}{12}\frac{1-\nu}{2} n^2 + \bar{\Omega}^2 - q_1 n^2$$

$$a_{12} = \frac{1+\nu}{2}\bar{\lambda} n$$

$$a_{13} = \nu\bar{\lambda} + \frac{\bar{\alpha}^2}{12}\left(\bar{\lambda}^3 - \frac{1-\nu}{2} n^2 \bar{\lambda}\right) - q_1 \bar{\lambda}$$

$$a_{21} = a_{12}$$

$$a_{22} = -n^2 - \frac{1-\nu}{2}\bar{\lambda}^2 - \frac{\bar{\alpha}^2}{12}\frac{3}{2}(1-\nu)\bar{\lambda}^2 - q_2\bar{\lambda}^2 + \bar{\Omega}^2$$

$$a_{23} = -n - \frac{\bar{\alpha}^2}{12}\frac{3-\nu}{2} n \bar{\lambda}^2$$

$$a_{31} = a_{13}$$

$$a_{32} = a_{23}$$

$$a_{33} = -1 - \frac{\bar{\alpha}^2}{12}\left(\bar{\lambda}^4 + 2\bar{\lambda}^2 n^2 + n^4 - 2n^2 + 1\right) + q_1(1-n^2) - q_2\bar{\lambda}^2 + \bar{\Omega}^2 + \chi' + \gamma'$$

$$b_{11} = -\Omega S$$

$$b_{22} = b_{11}$$

$$b_{33} = -\theta' - \bar{\Omega} S$$

where $\bar{\lambda} = 2\pi a/\lambda_m$, $\bar{\alpha} = h/a$, $\Omega = \omega a/c_p$, $c_p = \sqrt{\dfrac{E}{\rho_t(1-\nu^2)}}$,

$S = 2\bar{n}a/c_p = \kappa a/\rho h \; 1/c_p$, $q_1 = \dfrac{(p_{s_i} - p_{s_o})a(1-\nu^2)}{Eh}$

$q_2 = \dfrac{1}{2}\dfrac{(p_{s_i} - p_{s_o})a(1-\nu^2)}{Eh}$, $\chi' = \Omega \dfrac{\rho_o}{\rho_t}\dfrac{c_o}{c_p}\dfrac{a}{h}\chi_{mn}$,

$\theta' = \Omega \dfrac{\rho_c}{\rho_t}\dfrac{C_c}{C_p}\dfrac{a}{h}\theta_{mn}$, $\gamma' = \Omega^2 \dfrac{\rho_c}{\rho_t}\dfrac{a}{h}\gamma_{mn}$

θ_{mn} and χ_{mn} were given before. γ_{mn} is as follows:

If $\psi\, c_r/c_i > 1$ $\quad \gamma_{mn}(k_r a) = \dfrac{J_n(k_r a)}{k_r a\left[-J_{n+1}(k_r a) + \frac{n}{k_r a} J_n(k_r a)\right]}$

$\psi\, c_r/c_i < 1$ $\quad \gamma_{mn}(k_r' a) = \dfrac{I_n(k_r' a)}{k_r' a\left[I_{n+1}(k_r' a) + \frac{n}{k_r' a} I_n(k_r' a)\right]}$

The damping coefficient ς is determined from the formula

$$\varsigma = 2\bar{n}a/c_p \text{ (where } K = 2\bar{n}\rho h \text{ in equations of motion)}$$

where $\bar{n}/p = c/c_c$ (ratio of damping to critical damping)

We first compute the natural frequency p without damping, then assume a ratio of c/c_c, calculate $\bar{n}$ and then ς.

From equations [26] we can obtain the complex constants

$$A_{mn} = A_r + iA_i \quad , \quad B_{mn} = B_r + iB_i \quad , \quad C_{mn} = C_r + iC_i$$

where subscripts r and i denote the real and imaginary parts. The expressions for the displacements and fluid pressure can then be written as follows

$$u_{mn} = \sqrt{A_r^2 + A_i^2}\, \cos\frac{2\pi x}{\lambda_m} \cos n\varphi\, e^{i(\omega t - \varphi_{mn})}$$

$$v_{mn} = \sqrt{B_r^2 + B_i^2}\, \sin\frac{2\pi x}{\lambda_m} \sin n\varphi\, e^{i(\omega t - \varphi_{mn})}$$

$$w_{mn} = \sqrt{C_r^2 + C_i^2}\, \sin\frac{2\pi x}{\lambda_m} \cos n\varphi\, e^{i(\omega t - \varphi_{mn})}$$

$$(p_{fo})_{mn} = \omega\rho_o c_o \sqrt{(C_r \chi_{mn} + C_i \theta_{mn})^2 + (C_r \theta_{mn} - C_i \chi_{mn})^2}\, \sin\frac{2\pi x}{\lambda_m} \cos n\varphi\, e^{i(\omega t - \varphi_{mn})}$$

$$(p_{fi})_{mn} = \rho_i \omega^2 \gamma_{mn} w_{mn}$$

The formulas for the longitudinal and periphery stresses at the outer surface of the shell are as follows:

$$(\sigma_x)_{mn} = P_{mn}\frac{a}{h}\sqrt{\left(-\bar{\lambda}A_r + \nu n B_r + \bar{\alpha}\bar{\lambda}^2 C_r + \frac{\nu\bar{\alpha}}{1+\bar{\alpha}}n^2 C_r + \frac{\nu}{1+\bar{\alpha}}C_r\right)^2 + \left(-\bar{\lambda}A_i + \nu n B_i + \bar{\alpha}\bar{\lambda}^2 C_i + \frac{\nu\bar{\alpha}}{1+\bar{\alpha}}n^2 C_i + \frac{\nu}{1+\bar{\alpha}}C_i\right)^2} \quad [29]$$

$$(\sigma_\varphi)_{mn} = P_{mn}\frac{a}{h}\sqrt{\left(-\nu\lambda A_r + n B_r + \frac{\bar{\alpha}}{1+\bar{\alpha}}n^2 C_r + \frac{1}{1+\bar{\alpha}}C_r + \nu\bar{\alpha}\bar{\lambda}^2 C_r\right)^2 + \left(-\nu\lambda A_i + n B_i + \frac{\bar{\alpha}}{1+\bar{\alpha}}n^2 C_i + \frac{1}{1+\bar{\alpha}}C_i + \nu\bar{\alpha}\bar{\lambda}^2 C_i\right)^2}$$

Substituting the expressions for the displacements we can finally write the equations for the stresses as follows

$$(\sigma_x)_{mn} = \bar{\sigma}_x\, P_{mn} \sin\frac{2\pi x}{\lambda_m} \cos n\varphi\, e^{i(\omega t - \varphi_{mn})}$$

$$(\sigma_\varphi)_{mn} = \bar{\sigma}_\varphi\, P_{mn} \sin\frac{2\pi x}{\lambda_m} \cos n\varphi\, e^{i(\omega t - \varphi_{mn})} \quad [30]$$

The deflections, pressures and stresses can be written in terms of dimensionless quantities as before with the thick shell.

Assuming that the internal driving pressure is of the form given by eq. [25] the formulas are as follows:

$$u_{mn} = \bar{u}\,\frac{a^2(1-\nu^2)}{Eh}\,P_{mn}\cos\frac{2\pi x}{\lambda_m}\cos n\phi\, e^{i(\omega t-\phi_{mn})}$$

$$v_{mn} = \bar{v}\,\frac{a^2(1-\nu^2)}{Eh}\,P_{mn}\sin\frac{2\pi x}{\lambda_m}\sin n\phi\, e^{i(\omega t-\phi_{mn})}$$

$$w_{mn} = \bar{w}\,\frac{a^2(1-\nu^2)}{Eh}\,P_{mn}\sin\frac{2\pi x}{\lambda_m}\cos n\phi\, e^{i(\omega t-\phi_{mn})} \qquad [31]$$

$$(p_{fo})_{mn} = P_{mn}\,\bar{P}_o\sin\frac{2\pi x}{\lambda_m}\cos n\phi\, e^{i(\omega t-\phi_{mn})}$$

$$(p_{fi})_{mn} = P_{mn}\,\bar{P}_i\sin\frac{2\pi x}{\lambda_m}\cos n\phi\, e^{i(\omega t-\phi_{mn})}$$

If $\quad p_i = P_o\, f(\phi, z)\, e^{i\omega t}$

then $$P_{mn} = \frac{2P_o}{\pi\lambda_m}\int_0^{2\pi}\int_0^{\lambda_m} f(\phi, z)\sin\frac{2\pi x}{\lambda_m}\cos n\phi\, dz\, d\phi \qquad [32]$$

For any loading which is harmonic in time the total response will then be the sum of the modal contributions as explained before for the thick shell.

In the simplified theory we can determine the deflections fluid pressure and stresses as a function of frequency for the following input parameters:

1. Wave length parameter $\bar{\lambda}$
2. Circumferential parameter n
3. Thickness parameter $\bar{a}$
4. Poisson's ratio ν
5. Damping parameter S
6. Static pressure parameters q_1, q_2
7. Wave velocity parameters

   c_r/c_i, c_r/c_o, c_o/c_p
8. Density parameters

   ρ_o/ρ_t, ρ_i/ρ_t

Since the above solution is again equivalent to the solution for two harmonic waves propagating along the tube, we will also be able to study the effects of the input parameters on the propagation of unattenuated elastic waves in the tube wall or unattenuated pressure waves inside the tube.

IV. RESULTS

A. Correlations between thick and thin shell theory for shells in an acoustic medium

Fig. 4 gives comparisons between the exact elasticity theory and the approximate theory for shells vibrating under water. It is seen that the approximate theory is excellent for shells with a ratio of inside to outside radius of 0.9. Both the natural frequency and radial displacement are predicted very accurately by the approximate theory. However for a much thicker shell with $\alpha = 0.7$ the approximate theory is not accurate for displacement prediction. The approximate theory essentially imposes constraints on the shell since an apriori distribution through the thickness is assumed. Therefore the approximate theory predicts a stiffer shell with consequent higher natural frequency and smaller displacements. These characteristics are illustrated in Fig. 4 where it is seen that the resonant displacements predicted by the approximate theory can be in error by a factor of 2 for the thicker shells. The natural frequency on the other hand is predicted within several percent by the approximate theory.

B. Comparisons between natural frequencies in vacuum and in water

Fig. 5 presents plots of frequency parameter, $\bar{\Omega}$ as a function of longitudinal wave length parameter β, for various circumferential nodal patterns. For the thicker shells ($\alpha = 0.7$) it is seen that the water effects the natural frequency very little. For the thinner shells ($\alpha = 0.95$) the water does not effect the natural frequency for $n = 4$ as much as it does for the modes of lower n. The frequencies for the first branch of the axially symmetric ($n = 0$) mode at long wave lengths are unaffected by the water, since this type of mode is primarily longitudinal at long wave lengths. In the infinitely long shell water pressure only comes about by virtue of radial motion. The water does effect the second branch frequencies of thin shells at long wave lengths since they are radial modes giving rise to appreciable added mass of water. The beam mode ($n = 1$) and the lobar modes ($n \geq 2$) have a radial component of displacement at long wave lengths (small β) and therefore the natural frequencies in water are considerably effected.

C. Thick shells - higher branches and higher orders

In the thick shell theory, for each value of n and β there is an infinite number of roots. The first resonance defines the frequency of the first branch at the given n and β, the second resonance defines the frequency of the second branch, etc. In wave propagation analysis or in general forced vibration analysis the importance of these higher branches and higher orders is of significance. Tables 2, 3 and 4, and Fig. 6 give the deflection amplitude for several of the modes.

For radiating modes ($\theta_{mn} > 0$) it is seen that the higher orders ($n = 3,5$) correspond to much larger amplitudes for thinner shells ($\alpha = 0.50$, $\alpha = 0.70$). The second branch for a cylinder with $\alpha = .01$ $\beta = 0.8$, $n = 1$ shown in Table 2 corresponds to much higher amplitudes than the first branches for $n = 3$ and 5. Although not illustrated in the table, it has been found that this is also true for the next several branches of the almost solid cylinder. On the other hand for the shells with $\alpha = 0.50$ and $\alpha = 0.70$ the amplitude of the first radiating mode near resonance for $n = 3$ and 5 is of the same order of magnitude as the first radiating mode for $n = 1$. The first radiating mode for $n = 1$ $\beta = 0.8$ corresponds to the second branch.

D. Sound power generated and resulting stresses

The average sound power transmitted to the medium over one period can be written as follows:

$$(Power)_{Ave} = \frac{1}{T}\int_0^T\int_A p v \, dA \, dt \qquad [33]$$

where T = one period

p = pressure

v = velocity

A = area

Substituting the expressions for the pressure and velocity the following expression is obtained for the power transmitted by the mnth mode

$$(P_{ave})_{mn} = \int_0^T\int_A \left(\rho_o c_o i\omega w_{mn} \mathcal{J}_{mn} \cos n\theta \sin\frac{2\pi x}{\lambda} e^{i\omega t}\right)\left(\omega w_{mn} \cos n\theta \sin\frac{2\pi x}{\lambda} e^{i(\omega t - \phi_{mn})}\right) dA \, dt \qquad [34]$$

Integrating with respect to time

$$(P_{ave})_{mn} = \frac{1}{2}\int_A \rho_o c_o w_{mn}^2 \omega^2 \mathcal{J}_{mn} \cos^2 n\theta \sin^2\frac{2\pi x}{\lambda} \cos\phi_{mn} \, dA \qquad [35]$$

where $$\cos\phi_{mn} = \frac{\theta_{mn}}{\mathcal{J}_{mn}} \qquad [36]$$

so $$(P_{ave})_{mn} = \frac{1}{2}\int_A \rho_o c_o w_{mn}^2 \omega^2 \theta_{mn} \cos^2 n\theta \sin^2\frac{2\pi x}{\lambda} \, dA \qquad [37]$$

The average power transmitted to the medium can then be written as follows for the approximate shell theory:

For the axilally symmetric modes ($n = 0$)

$$(P_{ave})_{mo} = 2\frac{(\bar{w})^2}{\bar{\alpha}^2}\bar{\Omega}^2 \theta_{mo}\left[\frac{1}{2}\frac{\rho_o}{\rho_t} c_o \frac{1-\nu^2}{E} P_i^2 \frac{A}{4}\right]$$

and for the nonaxially symmetric modes

$$(P_{ave})_{mn} = \frac{(\bar{w})^2}{\bar{\alpha}^2}\bar{\Omega}^2 \theta_{mn}\left[\frac{1}{2}\frac{\rho_o}{\rho_t} c_o \frac{1-\nu^2}{E} P_i^2 \frac{A}{4}\right]$$

In either case the power transmitted can be written as

$$(P_{ave})_{mn} = \bar{F}\left[\frac{1}{2}\frac{\rho_o}{\rho_t} c_o \frac{1-\nu^2}{E} P_i^2 \frac{A}{4}\right]$$

where $\bar{F}$ is a factor depending on the mode and the other physical parameters of the shell and medium. The term in the brackets is independent of the mode and thickness of the shell. In the above formula

$\bar{w}$ = non dimensional radial deflection

$\bar{\alpha} = \frac{h}{a}$

$\bar{\Omega}$ = non dimensional frequency parameter

θ_{mn} = resistive impedance

ρ_o = density of medium

ρ_t = density of shell

c_o = velocity of sound in medium

ν = Poisson's ratio for shell material

E = modulus of elasticity of shell

P_i = internal forcing pressure

A = surface area of cylinder

Thus for a given shell material, a given surface area, and a given internal driving force the power will be proportional to $\bar{F}$.

The output power cannot be used solely as a measure of the radiating characteristics of a given mode since large powers can be obtained by using large driving forces, thereby inducing large stresses in the shell. The maximum stress induced in the shell can be written in terms of the internal oscillating pressure as follows:

$$\sigma_{max} = \bar{\sigma}_{max} P_i$$

where P_i is the internal oscillating pressure and $\bar{\sigma}_{max}$ is a nondimensional quantity which is independent of the driving force. The following ratio therefore is a good measure of the power-stress capabilities of the shell

$$\bar{R} = \frac{\bar{F}}{(\bar{\sigma}_{max})^2}$$

For a given size radiator of surface area A made of a given material of modulus E and Poisson ratio ν, the ratio $\bar{R}$ gives the power that can be transmitted into the medium for a particular mode with a given maximum stress induced in the shell. This ratio is tabulated in Table 5 for different modes of vibration.

The results of Table 5 indicate that the shell must be driven with very large forces in the lobar modes (n = 2,4) in order for these modes to radiate just a fraction of the power that is radiated by the axially symmetric modes (n = 0). The first branch axially symmetric mode is primarily longitudinal at long wave lengths (small β) and consequently radiation from the cylindrical surface takes place through Poisson coupling. The second branch is primarily radial at long wave lengths and is the most efficient radiating mode of a cylindrical shell. This latter type of motion can be achieved in a cylindrical transducer either by keeping the ends of the transducer open so that uniform pulsing can take place or by making the shell very long compared to its diameter so that β will be small. Simplified equations for such a radiator are derived in the next section.

Although the first branch resonances of flexural waves for n = 1 and n = 2 are not associated with any radiation, Table 5 and Fig. 7c show that the second and third branches give appreciable radiation. For these higher branches for n = 1, 2 the power stress ratios will be of the same order of magnitude as the radial mode (n = 0).

In using large steel radiators, the main difficulty is weight. A long steel radiator that would resonate at low frequencies would have to be huge. To resonate at 200 cps in the radial mode a steel radiator would have to be 27 feet in diameter. Therefore materials with lower sound velocities or methods to reduce the sound velocity must be sought.

E. Equations for a radially pulsing cylinder

Assuming that the pressure in the outside medium is equalized by the static pressure in the internal fluid the equation of motion of the purely radial mode of a shell can be written as follows: (see Eq. [17]):

$$\frac{\rho_t a^2(1-\nu^2)}{E}\frac{\partial^2 w}{\partial t^2} + \frac{a^2(1-\nu^2)}{Eh}\kappa\frac{\partial w}{\partial t} - \frac{a^2(1-\nu^2)}{Eh}(p_{f_o} - p_{f_i}) + w + \frac{h^2}{12a^2}w = \frac{Pa^2(1-\nu^2)}{Eh}$$

Substituting the expressions for the external and internal fluid pressure due to sinusoidal radial pulsations of the cylinder the equation of motion becomes

$$\ddot{w} + \frac{\left[\frac{a^2(1-\nu^2)}{Eh}\kappa + \rho_o c_o \frac{a^2(1-\nu^2)}{Eh}\theta_{oo}\right]}{\left[\frac{\rho_t a^2(1-\nu^2)}{E} + \rho_i a\frac{a^2(1-\nu^2)}{Eh}\gamma_{oo} + \rho_o c_o\frac{a^2(1-\nu^2)}{Eh}\frac{\chi_{oo}}{\omega}\right]}\dot{w}$$

$$+ \frac{\left[1 + \frac{h^2}{12a^2}\right]}{\left[\frac{\rho_t a^2(1-\nu^2)}{E} + \rho_i a\frac{a^2(1-\nu^2)}{Eh}\gamma_{oo} + \rho_o c_o\frac{a^2(1-\nu^2)}{Eh}\frac{\chi_{oo}}{\omega}\right]}w = \frac{\frac{Pa^2(1-\nu^2)}{Eh}}{\left[\frac{\rho_t a^2(1-\nu^2)}{Eh} + \rho_i a\frac{a^2(1-\nu^2)}{Eh}\gamma_{oo} + \rho_o c_o\frac{a^2(1-\nu^2)}{Eh}\frac{\chi_{oo}}{\omega}\right]}$$

Letting $w = Ce^{i\omega t}$ and solving

$$C = \frac{Pa^2(1-\nu^2)/Eh}{\left[1+\frac{h^2}{12a^2} - \bar{\Omega}^2\left(1+\frac{\rho_i}{\rho_t}\frac{a}{h}\gamma_{00} + \frac{\rho_o}{\rho_t}\frac{c_o}{c_p}\frac{a}{h}\frac{\chi_{oo}}{\bar{\Omega}}\right)\right] + i\bar{\Omega}\left[\frac{\rho_o}{\rho_t}\frac{c_o}{c_p}\frac{a}{h}\theta_{00} + \frac{2\bar{n}a}{c_p}\right]}$$

where

$$\gamma_{00}\left(\frac{\omega}{c_i}a\right) = \frac{J_0\left(\frac{\omega}{c_i}a\right)}{\frac{\omega}{c_i}a\,J_1\left(\frac{\omega}{c_i}a\right)} = \frac{J_0(k'a)}{k'a\,J_1(k'a)} \qquad k'a = \frac{\omega a}{c_i}$$

and using the plane wave approximation[11]

$$\chi_{00}\left(\frac{\omega}{c_o}a\right) \approx \frac{(\tfrac{1}{2}ka)}{1+(\tfrac{1}{2}ka)^2} \qquad ka = \frac{\omega a}{c_o}$$

$$\theta_{00}\left(\frac{\omega}{c_o}a\right) \approx \frac{1}{1+(\tfrac{1}{2}ka)^2}$$

The numerator in the equation for C is the static deflection under a static pressure P so that C takes the form of the standard resonance factor for a single degree of freedom system.

The natural frequency is determined from the equation

$$\left(1+\frac{h^2}{12a^2}\right) - \bar{\Omega}^2\left(1+\frac{\rho_i}{\rho_t}\frac{a}{h}\gamma_{00} + \frac{\rho_o}{\rho_t}\frac{c_o}{c_p}\frac{a}{h}\frac{\chi_{00}}{\bar{\Omega}}\right) = 0$$

The values of $\bar{\Omega}$ which satisfy the above equation determine the natural frequencies of the system.

The Q of the system can be written as follows:

$$Q = \frac{\omega_{res}\left[\frac{\rho_t a^2(1-\nu^2)}{E} + \rho_i a\frac{a^2(1-\nu^2)}{Eh}\gamma_{00} + \rho_o c_o\frac{a^2(1-\nu^2)}{Eh}\frac{\chi_{00}}{\omega}\right]}{\frac{a^2(1-\nu^2)}{Eh}K + \rho_o c_o\theta_{00}\frac{a^2(1-\nu^2)}{Eh}}$$

using the frequency equation

$$Q = \frac{1+\frac{h^2}{12a^2}}{\omega_{res}\left(\frac{a^2(1-\nu^2)}{Eh}K + \rho_o c_o\theta_{00}\frac{a^2(1-\nu^2)}{Eh}\right)}$$

If δ = logarithmic decrement for structural vibration of the shell material, then

$$Q = \frac{1+\frac{h^2}{12a^2}}{\bar{\Omega}^2\frac{\delta}{2\pi} + \bar{\Omega}\frac{\rho_o}{\rho_t}\frac{c_o}{c_p}\frac{a}{h}\theta_{00}}$$

If $\frac{h}{a} < 0.1$ then to a very close approximation

$$\frac{1}{Q} = \bar{\Omega}^2\frac{\delta}{2\pi} + \bar{\Omega}\frac{\rho_o}{\rho_t}\frac{c_o}{c_p}\frac{a}{h}\theta_{00}$$

The efficiency of the radiator is as follows:

$$\text{Efficiency} = \frac{\bar{\Omega}^2\frac{\delta}{2\pi} + \bar{\Omega}\frac{\rho_o}{\rho_t}\frac{c_o}{c_p}\frac{a}{h}\theta_{00}}{2\bar{\Omega}^2\frac{\delta}{2\pi} + \bar{\Omega}\frac{\rho_o}{\rho_t}\frac{c_o}{c_p}\frac{a}{h}\theta_{00}}$$

or using the approximate formula for the radiation impedance,

11. Hueter, T.F. and Bolt, R.H., "Sonics," John Wiley & Sons, 1955, p. 58.

$$\text{Efficiency} \approx \frac{1 + \frac{\rho_0}{\rho_t}\frac{a}{h}\frac{4ka}{(2ka)^2+1}\frac{2\pi}{\delta}}{2 + \frac{\rho_0}{\rho_t}\frac{a}{h}\frac{4ka}{(2ka)^2+1}\frac{2\pi}{\delta}}$$

At the present time very little information is available on the structural damping of new high strength light weight plastics.

A study of these equations and the possible use of different materials will therefore be the subject of a future study, however a very rough estimate of the order of magnitude of the Q and the efficiency of a steel and plexiglass radiator is made below.

Using the thin shell theory derived in this report, it was found that at long wave lengths the radial mode of a steel radiator (air filled) with a/a_0=0.95 had its resonance at $\bar{\Omega} \approx 1$. Using a logarithmic decrement of 0.02 and ρ_0/ρ_t =0.127, c_0/c_p =0.278, $\frac{a}{h}$ = 20, the Q and efficiency are as follows (neglecting any other losses beside internal structural damping):

$$Q = \frac{1}{\frac{.02}{6.28} + 1 \times .127 \times .278 \times 20 \times 1} \approx 1.4$$

Efficiency ≈ 100%

For a plexiglass radiator with ρ_0/ρ_t =0.85, c_0/c_p =0.93, $\frac{a}{h}$ = 10 the resonance occurred at $\bar{\Omega}$ =0.2 with a value of ϑ =0.32. Assuming 100 times the damping in plexiglass as in steel

$$Q = \frac{1}{\frac{2}{6.28} + .2 \times .85 \times .93 \times 10 \times .32} \approx 1.15$$

Efficiency ≈ 70 %

In spite of the comparatively lower efficiency of the plexiglass radiator it should be noted that in order for the steel radiator to resonate at 200 cps it would have to be about 27 feet in diameter while the plexiglass radiator would be about 3 feet in diameter.

For the plexiglass radiator it was found that

$$\bar{F}/(\bar{\sigma}_{max})^2 \approx \frac{6.2}{(13.7)^2} = 0.03$$

For the steel radiator

$$\bar{F}/(\bar{\sigma}_{max})^2 \approx \frac{1652}{(29.2)^2} = 1.9$$

However since the modulii and mass ratio of steel and plastic are different the ratio of $P_{ave}/(\sigma_{max})^2$ must be taken instead of $\bar{F}/(\bar{\sigma}_{max})^2$

$$\frac{\left[\frac{P_{ave}}{(\sigma_{max})^2}\right]_{steel}}{\left[\frac{P_{ave}}{(\sigma_{max})^2}\right]_{plexiglass}} \approx 0.9\,\frac{A_s}{A_p}$$

A_s = surface area of steel radiator

A_p = surface area of plexiglass radiator

So that for the same area the power stress ratio is almost the same for the two materials. The main advantage of the steel is that it has much greater stress capability and therefore much greater power capability.

Taking the safe alternating stress in the plexiglass to be 2000 psi the power output for this working stress would be

$$P_i = 146$$

$$\therefore P_{ave} = 6.2\left[.5 \times .85 \times \frac{60,000 \times (146)^2 \times .91}{0.4 \times 10^6} \times \frac{A_p}{4}\right]$$

$$\approx 1916\ A_p\ \frac{\#\,in}{sec}\ (A_p \text{ in sq. in.})$$

Taking the safe alternating stress in the steel to be 20,000 psi

$$P_i = 685$$

$$\therefore P_{ave} = 1652\left[.5 \times \frac{.127 \times 60,000 \times (685)^2 \times .91}{30 \times 10^6} \times \frac{A_s}{4}\right]$$

$$= 22,400\ A_s$$

The steel is thus capable of delivering ten times the power as the plastic, however the size and consequent cost of the steel radiator is the actual drawback.

If the plexiglass radiator were 20 feet long and 3 feet in diameter it would have the capability of delivering the following power:

$$P_{ave} = 1916 \times \pi d \ell$$

$$= 1916 \times 3.14 \times 36 \times 240 = 51,978,000\ \frac{\#\,in}{sec}$$

$$\approx 5800 \text{ Kilowatts}$$

If the working stress were cut by 10 the power would be cut by 100. However this would still give about 60 KW. This indicates that a plastic radiator could conceivably be used as a high power low frequency sound source although much more careful study is needed before an actual design can be made since many important factors have been left out in the foregoing analysis.

F. Some effects of internal fluid

Fig. 7a and 7b give some typical response curves including the effects of internal pressure and internal fluid. It is seen that the internal pressure and fluid have a larger effect on the lobar mode frequency (n=2) than on the axially symmetric mode (n=0). This general effect of pressure is also shown by Baron and Bleich[2] in their more extensive calculations of the effects of internal pressure. It is also illustrated in Fig. 7b that the internal pressure tends to stiffen the shell thus decreasing the static deflection for a given driving pressure.

For the internal fluid considered here the natural frequencies of both the n = 0 and n = 2 modes were decreased indicating that for these frequencies the internal fluid has a positive reactance. Other cases can exist where the fluid has the opposite effect.

In general it can be stated that unless the shell is extremely thin ($h/a < 0.01$) the internal fluid or pressure will only have a small effect on the frequencies. It was pointed out to the author that Dr. G. B. Warburton had found similar results for thin shells which he reported orally at the Stresa Conference several months ago.

G. Electronic computer codes available for calculation

This report contains only a small number of the results that have been computed. It has been the purpose of the report to present the basic theory and some general trends giving the effect of some of the physical parameter.

IBM 709 codes are available for computing the response curves for thick shells vibrating in any fluid. This code is based on the exact elasticity theory presented here. For this code the computer tabulates the radial displacement, the resistive and reactive impedance and the external pressure for any given driving frequency.

Codes are also available for computing the response curves of thin pressurized shells containing fluid. For this case the computer prints out the axial, tangential, and radial displacements; the internal and external pressure; the resistive and reactive impedance; and the longitudinal and tangential stresses. For the exact theory it takes about four seconds to calculate the response at one frequency. The approximate theory caculuations take about one-half second for each frequency.

Using these codes together with the general relations presented in the earlier part of this report one can compute the forced response of thick and thin cylindrical shells vibrating in an acoustic medium.

APPENDIX I

I. The values for the constants $a_{11} \ldots a_{66}$ which are contained in the body of the report are as follows:

A. If $\omega/C_d < 2\pi/\lambda_m$, $\omega/C_r < 2\pi/\lambda_m$ then $\rho = \beta\sqrt{1-\psi^2}$, $\psi = \frac{\Omega}{\beta}\sqrt{\frac{2}{1-\nu}}$, $\eta = \left\{\frac{[\beta^2 + (1-2\nu)\rho^2]}{2(1-\nu)}\right\}^{1/2}$ and

$$a_{11} = \left[n^2+n+\eta^2 - \frac{\nu}{1-2\nu}(\beta^2-\eta^2)\right] I_n(\eta) - \eta I_{n-1}(\eta)$$

$$a_{12} = \left[n^2+n+\eta^2 - \frac{\nu}{1-2\nu}(\beta^2-\eta^2)\right] K_n(\eta) + \eta K_{n-1}(\eta)$$

$$a_{13} = [n^2+n+\rho^2] I_n(\rho) - \rho I_{n-1}(\rho)$$

$$a_{14} = [n^2+n+\rho^2] K_n(\rho) + \rho K_{n-1}(\rho)$$

$$a_{15} = -(n+1) I_n(\rho) + \rho I_{n-1}(\rho)$$

$$a_{16} = -(n+1) K_n(\rho) - \rho K_{n-1}(\rho)$$

$$a_{21} = \left[n^2+n+\alpha^2\eta^2 - \frac{\nu\alpha^2}{1-2\nu}(\beta^2-\eta^2)\right] I_n(\alpha\eta) - \alpha\eta I_{n-1}(\alpha\eta)$$

$$a_{22} = \left[n^2+n+\alpha^2\eta^2 - \frac{\nu\alpha^2}{1-2\nu}(\beta^2-\eta^2)\right] K_n(\alpha\eta) + \alpha\eta K_{n-1}(\alpha\eta)$$

$$a_{23} = [n^2+n+\alpha^2\rho^2] I_n(\alpha\rho) - \alpha\rho I_{n-1}(\alpha\rho)$$

$$a_{24} = [n^2+n+\alpha^2\rho^2] K_n(\alpha\rho) + \alpha\rho K_{n-1}(\alpha\rho)$$

$$a_{25} = \alpha\rho I_{n-1}(\alpha\rho) - (n+1) I_n(\alpha\rho)$$

$$a_{26} = -\alpha\rho K_{n-1}(\alpha\rho) - (n+1) K_n(\alpha\rho)$$

$$a_{31} = (n+1) I_n(\eta) - \eta I_{n-1}(\eta)$$

$$a_{32} = (n+1) K_n(\eta) + \eta K_{n-1}(\eta)$$

$$a_{33} = (n+1) I_n(\rho) - \rho I_{n-1}(\rho)$$

$$a_{34} = (n+1) K_n(\rho) + \rho K_{n-1}(\rho)$$

$$a_{35} = -\left(\frac{1}{n}+1+\frac{\rho^2}{2n^2}\right) I_n(\rho) + \frac{\rho}{n^2} I_{n-1}(\rho)$$

$$a_{36} = -\left(\frac{1}{n}+1+\frac{\rho^2}{2n^2}\right) K_n(\rho) - \frac{\rho}{n^2} K_{n-1}(\rho)$$

$$a_{41} = (n+1) I_n(\alpha\eta) - \alpha\eta I_{n-1}(\alpha\eta)$$

$$a_{42} = (n+1) K_n(\alpha\eta) + \alpha\eta K_{n-1}(\alpha\eta)$$

$$a_{43} = (n+1) I_n(\alpha\rho) - \alpha\rho I_{n-1}(\alpha\rho)$$

$$a_{44} = (n+1) K_n(\alpha\rho) + \alpha\rho K_{n-1}(\alpha\rho)$$

$$a_{45} = -\left(\frac{1}{n}+1+\frac{\alpha^2\rho^2}{2n^2}\right) I_n(\alpha\rho) + \frac{\alpha\rho}{n^2} I_{n-1}(\alpha\rho)$$

$$a_{46} = -\left(\frac{1}{n}+1+\frac{\alpha^2\rho^2}{2n^2}\right) K_n(\alpha\rho) - \frac{\alpha\rho}{n^2} K_{n-1}(\alpha\rho)$$

$$a_{51} = \eta I_{n-1}(\eta) - n I_n(\eta)$$

$$a_{52} = -[\eta K_{n-1}(\eta) + n K_n(\eta)]$$

$$a_{53} = \frac{1}{2}\left[1+\frac{\rho^2}{\beta^2}\right]\left[\rho I_{n-1}(\rho) - n I_n(\rho)\right]$$

$$a_{54} = \frac{1}{2}\left[1+\frac{\rho^2}{\beta^2}\right]\left[-\rho K_{n-1}(\rho) - n K_n(\rho)\right]$$

$$a_{55} = \frac{I_n(\rho)}{2}$$

$$a_{56} = \frac{K_n(\rho)}{2}$$

$$a_{61} = \alpha\eta I_{n-1}(\alpha\eta) - n I_n(\alpha\eta)$$

$$a_{62} = -[\alpha\eta K_{n-1}(\alpha\eta) + n K_n(\alpha\eta)]$$

$$a_{63} = \frac{1}{2}\left[1+\frac{\rho^2}{\beta^2}\right]\left[\alpha\rho I_{n-1}(\alpha\rho) - n I_n(\alpha\rho)\right]$$

$$a_{64} = \frac{1}{2}\left[1+\frac{\rho^2}{\beta^2}\right]\left[-\alpha\rho K_{n-1}(\alpha\rho) - n K_n(\alpha\rho)\right]$$

$$a_{65} = \frac{I_n(\alpha\rho)}{2}$$

$$a_{66} = \frac{K_n(\alpha\rho)}{2}$$

B. If $\omega/c_d < 2\pi/\lambda_m$, $\omega/c_r > 2\pi/\lambda_m$ then

$$\zeta = \beta\sqrt{\psi^2 - 1} \qquad \psi = \frac{\Omega}{\beta}\sqrt{\frac{2}{1-\nu}}\,, \quad \eta = \left\{\frac{[\beta^2 - (1-2\nu)\zeta^2]}{2(1-\nu)}\right\}^{1/2}$$

and

$$a_{11} = \left[n^2 + n + \eta^2 - \frac{\nu}{1-2\nu}(\beta^2 - \eta^2)\right] I_n(\eta) - \eta I_{n-1}(\eta)$$
$$a_{12} = \left[n^2 + n + \eta^2 - \frac{\nu}{1-2\nu}(\beta^2 - \eta^2)\right] K_n(\eta) + \eta K_{n-1}(\eta)$$
$$a_{13} = [n^2 + n - \zeta^2] J_n(\zeta) - \zeta J_{n-1}(\zeta)$$
$$a_{14} = [n^2 + n - \zeta^2] Y_n(\zeta) - \zeta Y_{n-1}(\zeta)$$
$$a_{15} = \zeta J_{n-1}(\zeta) - (n+1) J_n(\zeta)$$
$$a_{16} = \zeta Y_{n-1}(\zeta) - (n+1) Y_n(\zeta)$$
$$a_{21} = \left[n^2 + n + \alpha^2\eta^2 - \frac{\nu\alpha^2}{1-2\nu}(\beta^2 - \eta^2)\right] I_n(\alpha\eta) - \alpha\eta I_{n-1}(\alpha\eta)$$
$$a_{22} = \left[n^2 + n + \alpha^2\eta^2 - \frac{\nu\alpha^2}{1-2\nu}(\beta^2 - \eta^2)\right] K_n(\alpha\eta) + \alpha\eta K_{n-1}(\alpha\eta)$$
$$a_{23} = [n^2 + n - \alpha^2\zeta^2] J_n(\alpha\zeta) - \alpha\zeta J_{n-1}(\alpha\zeta)$$
$$a_{24} = [n^2 + n - \alpha^2\zeta^2] Y_n(\alpha\zeta) - \alpha\zeta Y_{n-1}(\alpha\zeta)$$
$$a_{25} = \alpha\zeta J_{n-1}(\alpha\zeta) - (n+1) J_n(\alpha\zeta)$$
$$a_{26} = \alpha\zeta Y_{n-1}(\alpha\zeta) - (n+1) Y_n(\alpha\zeta)$$
$$a_{31} = (n+1) I_n(\eta) - \eta I_{n-1}(\eta)$$
$$a_{32} = (n+1) K_n(\eta) + \eta K_{n-1}(\eta)$$
$$a_{33} = (n+1) J_n(\zeta) - \zeta J_{n-1}(\zeta)$$
$$a_{34} = (n+1) Y_n(\zeta) - \zeta Y_{n-1}(\zeta)$$
$$a_{35} = \frac{\zeta}{n^2} J_{n-1}(\zeta) - \left(1 + \frac{1}{n} - \frac{\zeta^2}{2n^2}\right) J_n(\zeta)$$
$$a_{36} = \frac{\zeta}{n^2} Y_{n-1}(\zeta) - \left(1 + \frac{1}{n} - \frac{\zeta^2}{2n^2}\right) Y_n(\zeta)$$

$$a_{41} = (n+1) I_n(\alpha\eta) - \alpha\eta I_{n-1}(\alpha\eta)$$
$$a_{42} = (n+1) K_n(\alpha\eta) + \alpha\eta K_{n-1}(\alpha\eta)$$
$$a_{43} = (n+1) J_n(\alpha\zeta) - \alpha\zeta J_{n-1}(\alpha\zeta)$$
$$a_{44} = (n+1) Y_n(\alpha\zeta) - \alpha\zeta Y_{n-1}(\alpha\zeta)$$
$$a_{45} = \frac{\alpha\zeta}{n^2} J_{n-1}(\alpha\zeta) - \left(1 + \frac{1}{n} - \frac{\alpha^2\zeta^2}{2n^2}\right) J_n(\alpha\zeta)$$
$$a_{51} = \eta I_{n-1}(\eta) - n I_n(\eta)$$
$$a_{52} = -[\eta K_{n-1}(\eta) + n K_n(\eta)]$$
$$a_{53} = \frac{1}{2}\left[1 - \frac{\zeta^2}{\beta^2}\right]\left[\zeta J_{n-1}(\zeta) - n J_n(\zeta)\right]$$
$$a_{54} = \frac{1}{2}\left[1 - \frac{\zeta^2}{\beta^2}\right]\left[\zeta Y_{n-1}(\zeta) - n Y_n(\zeta)\right]$$
$$a_{55} = \frac{J_n(\zeta)}{2}$$
$$a_{56} = \frac{Y_n(\zeta)}{2}$$
$$a_{61} = \alpha\eta I_{n-1}(\alpha\eta) - n I_n(\alpha\eta)$$
$$a_{52} = -[\alpha\eta K_{n-1}(\alpha\eta) + n K_n(\alpha\eta)]$$
$$a_{63} = \frac{1}{2}\left[1 - \frac{\zeta^2}{\beta^2}\right]\left[\alpha\zeta J_{n-1}(\alpha\zeta) - n J_n(\alpha\zeta)\right]$$
$$a_{64} = \frac{1}{2}\left[1 - \frac{\zeta^2}{\beta^2}\right]\left[\alpha\zeta Y_{n-1}(\alpha\zeta) - n Y_n(\alpha\zeta)\right]$$
$$a_{65} = \frac{J_n(\alpha\zeta)}{2}$$
$$a_{66} = \frac{Y_n(\alpha\zeta)}{2}$$

C. If $\omega/c_d > 2\pi/\lambda_m$, $\omega/c_r > 2\pi/\lambda_m$

$$\zeta = \beta\sqrt{\psi^2 - 1}\,,\quad \psi = \frac{\Omega}{\beta}\sqrt{\frac{2}{1-\nu}}\,,\quad \eta = \left\{\frac{\left[(1-2\nu)\zeta^2 - \beta^2\right]}{2(1-\nu)}\right\}^{1/2}$$

and

$$a_{11} = \left[n^2+n-\eta^2 - \frac{\nu}{1-2\nu}(\beta^2+\eta^2)\right] J_n(\eta) - \eta J_{n-1}(\eta)$$

$$a_{12} = \left[n^2+n-\eta^2 - \frac{\nu}{1-2\nu}(\beta^2+\eta^2)\right] Y_n(\eta) - \eta Y_{n-1}(\eta)$$

$$a_{13} = \left[n^2+n-\zeta^2\right] J_n(\zeta) - \zeta J_{n-1}(\zeta)$$

$$a_{14} = \left[n^2+n-\zeta^2\right] Y_n(\zeta) - \zeta Y_{n-1}(\zeta)$$

$$a_{15} = \zeta J_{n-1}(\zeta) - (n+1) J_n(\zeta)$$

$$a_{16} = \zeta Y_{n-1}(\zeta) - (n+1) Y_n(\zeta)$$

$$a_{21} = \left[n^2+n-\alpha^2\eta^2 - \frac{\nu\alpha^2}{1-2\nu}(\beta^2+\eta^2)\right] J_n(\alpha\eta) - \alpha\eta J_{n-1}(\alpha\eta)$$

$$a_{22} = \left[n^2+n-\alpha^2\eta^2 - \frac{\nu\alpha^2}{1-2\nu}(\beta^2+\eta^2)\right] Y_n(\alpha\eta) - \alpha\eta Y_{n-1}(\alpha\eta)$$

$$a_{23} = \left[n^2+n-\alpha^2\zeta^2\right] J_n(\alpha\zeta) - \alpha\zeta J_{n-1}(\alpha\zeta)$$

$$a_{24} = \left[n^2+n-\alpha^2\zeta^2\right] Y_n(\alpha\zeta) - \alpha\zeta Y_{n-1}(\alpha\zeta)$$

$$a_{25} = \alpha\zeta J_{n-1}(\alpha\zeta) - (n+1) J_n(\alpha\zeta)$$

$$a_{26} = \alpha\zeta Y_{n-1}(\alpha\zeta) - (n+1) Y_n(\alpha\zeta)$$

$$a_{31} = (n+1) J_n(\eta) - \eta J_{n-1}(\eta)$$

$$a_{32} = (n+1) Y_n(\eta) - \eta Y_{n-1}(\eta)$$

$$a_{33} = (n+1) J_n(\zeta) - \zeta J_{n-1}(\zeta)$$

$$a_{34} = (n+1) Y_n(\zeta) - \zeta Y_{n-1}(\zeta)$$

$$a_{35} = \frac{\zeta}{n^2} J_{n-1}(\zeta) - \left(1 + \frac{1}{n} - \frac{\zeta^2}{2n^2}\right) J_n(\zeta)$$

$$a_{36} = \frac{\zeta}{n^2} Y_{n-1}(\zeta) - \left(1 + \frac{1}{n} - \frac{\zeta^2}{2n^2}\right) Y_n(\zeta)$$

$$a_{41} = (n+1) J_n(\alpha\eta) - \alpha\eta J_{n-1}(\alpha\eta)$$

$$a_{42} = (n+1) Y_n(\alpha\eta) - \alpha\eta Y_{n-1}(\alpha\eta)$$

$$a_{43} = (n+1) J_n(\alpha\zeta) - \alpha\zeta J_{n-1}(\alpha\zeta)$$

$$a_{44} = (n+1) Y_n(\alpha\zeta) - \alpha\zeta Y_{n-1}(\alpha\zeta)$$

$$a_{45} = \frac{\alpha\zeta}{n^2} J_{n-1}(\alpha\zeta) - \left(1 + \frac{1}{n} - \frac{\alpha^2\zeta^2}{2n^2}\right) J_n(\alpha\zeta)$$

$$a_{46} = \frac{\alpha\zeta}{n^2} Y_{n-1}(\alpha\zeta) - \left(1 + \frac{1}{n} - \frac{\alpha^2\zeta^2}{2n^2}\right) Y_n(\alpha\zeta)$$

$$a_{51} = \eta J_{n-1}(\eta) - n J_n(\eta)$$

$$a_{52} = \eta Y_{n-1}(\eta) - n Y_n(\eta)$$

$$a_{53} = \frac{1}{2}\left[1 - \frac{\zeta^2}{\beta^2}\right]\left[\zeta J_{n-1}(\zeta) - n J_n(\zeta)\right]$$

$$a_{54} = \frac{1}{2}\left[1 - \frac{\zeta^2}{\beta^2}\right]\left[\zeta Y_{n-1}(\zeta) - n Y_n(\zeta)\right]$$

$$a_{55} = \frac{J_n(\zeta)}{2}$$

$$a_{56} = \frac{Y_n(\zeta)}{2}$$

$$a_{61} = \alpha\eta J_{n-1}(\alpha\eta) - n J_n(\alpha\eta)$$

$$a_{62} = \alpha\eta Y_{n-1}(\alpha\eta) - n Y_n(\alpha\eta)$$

$$a_{63} = \frac{1}{2}\left[1 - \frac{\zeta^2}{\beta^2}\right]\left[\alpha\zeta J_{n-1}(\alpha\zeta) - n J_n(\alpha\zeta)\right]$$

$$a_{64} = \frac{1}{2}\left[1 - \frac{\zeta^2}{\beta^2}\right]\left[\alpha\zeta Y_{n-1}(\alpha\zeta) - n Y_n(\alpha\zeta)\right]$$

$$a_{65} = \frac{J_n(\alpha\zeta)}{2}$$

$$a_{66} = \frac{Y_n(\alpha\zeta)}{2}$$

II. The expressions for the resistive and reactive components of impedance to be used with the exact elasticity theory are as follows:

For $\psi \, {}^{c_r}/_{c_o} > 1$

$$\chi_{mn} = \frac{\beta\psi \, {}^{c_r}/_{c_o}\left\{J_n(k_m a_o)\left[J_{n-1}(k_m a_o) - \frac{n}{k_m a_o} J_n(k_m a_o)\right] + Y_n(k_m a_o)\left[Y_{n-1}(k_m a_o) - \frac{n}{k_m a_o} Y_n(k_m a_o)\right]\right\}}{k_m a_o\left\{\left[J_{n-1}(k_m a_o) - \frac{n}{k_m a_o} J_n(k_m a_o)\right]^2 + \left[Y_{n-1}(k_m a_o) - \frac{n}{k_m a_o} Y_n(k_m a_o)\right]^2\right\}}$$

$$\theta_{mn} = \frac{-2\beta\psi \, {}^{c_r}/_{c_o}}{\pi(k_m a_o)^2\left\{\left[J_{n-1}(k_m a_o) - \frac{n}{k_m a_o} J_n(k_m a_o)\right]^2 + \left[Y_{n-1}(k_m a_o) - \frac{n}{k_m a_o} Y_n(k_m a_o)\right]^2\right\}}$$

where $k_m a_o = \sqrt{(\beta\psi \, {}^{c}/_{c_o})^2 - \beta^2}$

For $\psi \, {}^{c_r}/_{c_o} < 1$

$$\chi_{mn} = \frac{-\beta\psi \, {}^{c_r}/_{c_o} \, K_n(k_m' a_o)}{k_m' a_o\left[K_{n-1}(k_m' a_o) + \frac{n}{k_m' a_o} K_n(k_m' a_o)\right]}$$

$$\theta_{mn} = 0$$

where $k_m' a_o = \sqrt{\beta^2 - (\beta\psi \, {}^{c}/_{c_o})^2}$

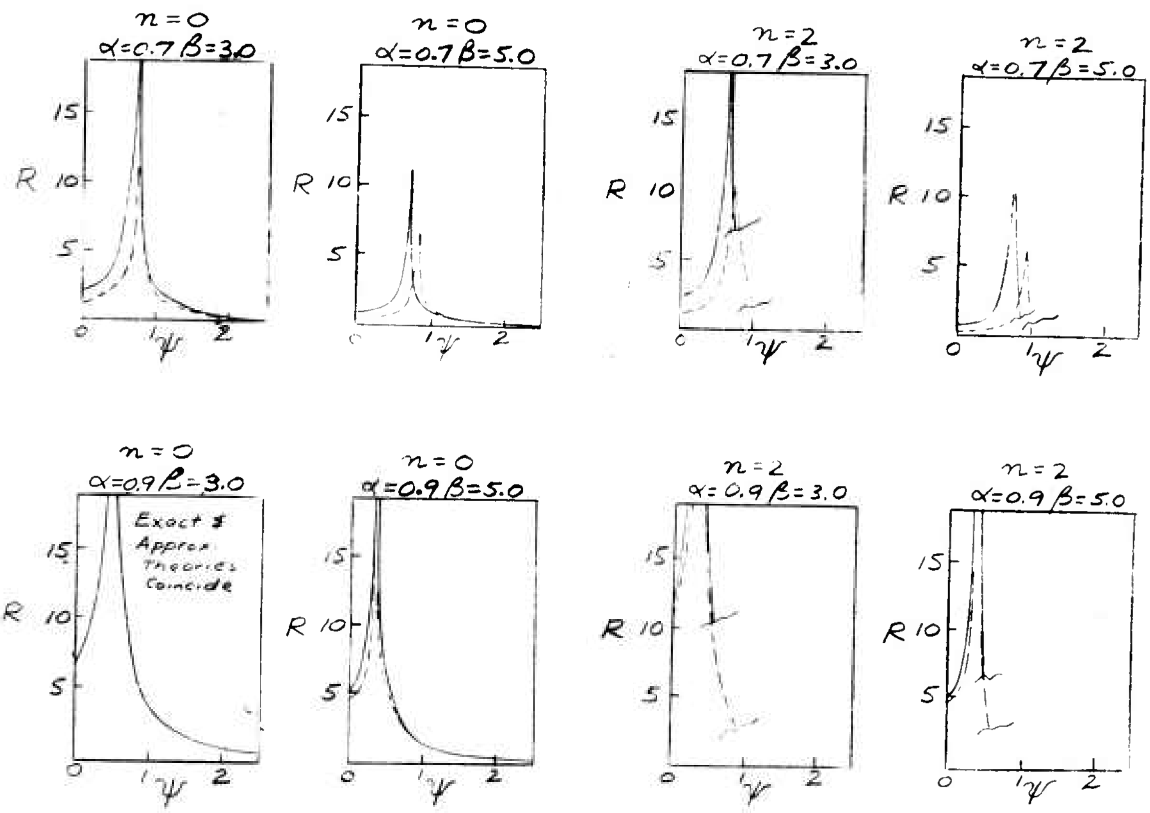


Fig. 4 Comparisons – Exact and Approximate Shell Theories
(Water Outside, Air Inside; $c_o/c_r = 0.47$, $\rho_o/\rho = 0.13$, $\nu = 0.3$)
——— Exact Theory - - - - Flügge Shell Theory

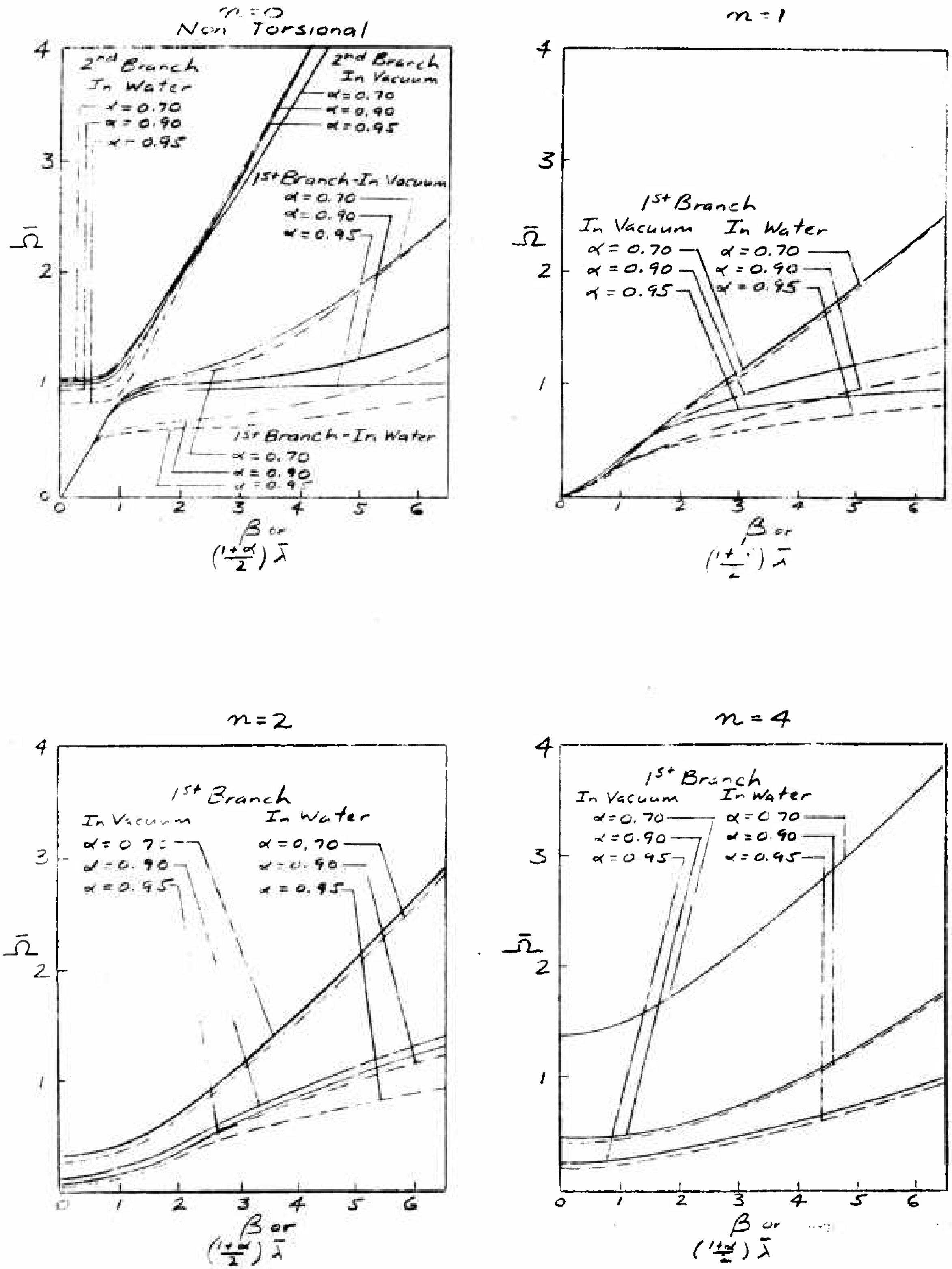


Fig. 5 Frequency in Vacuum and in Water
(Approximate Shell Theory - Flügge)
$c_o/c_r = 0.47$, $\rho_o/\rho = 0.13$, $\nu = 0.3$

Fig. 6 Resonance Curves from Exact Theory, $\beta = 3.0$ ($c_0/c_1 = 0.47$, $\rho_0/\rho = 0.13$, $\nu = 0.3$)

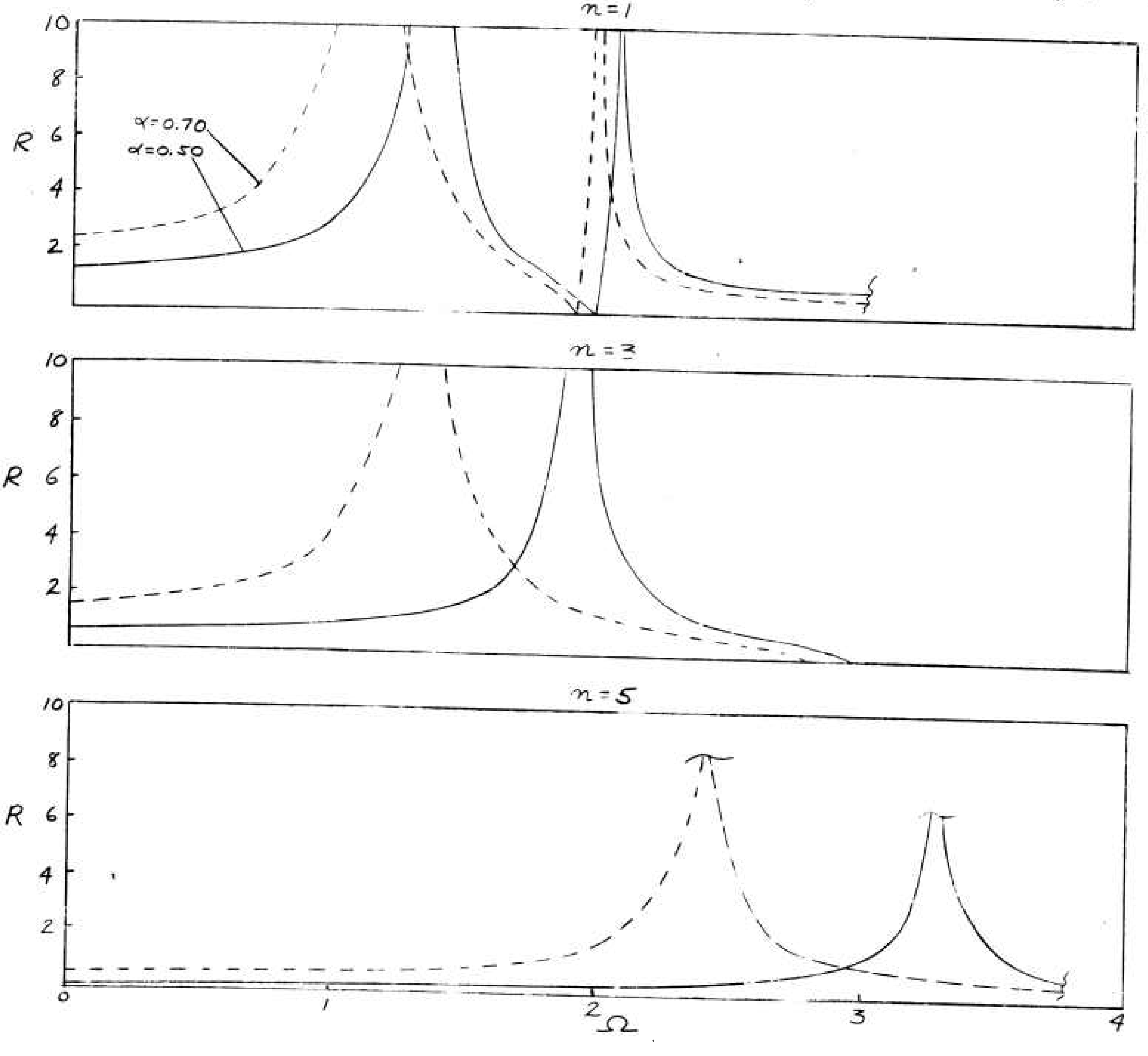


Fig. 7a. Effect of Internal and External Fluid (ν=0.3, n=0, α=0.95, β=0.10) ρ_{air}/ρ_{tube}=0.00016 ρ_{oil}/ρ_{tube}=0.090

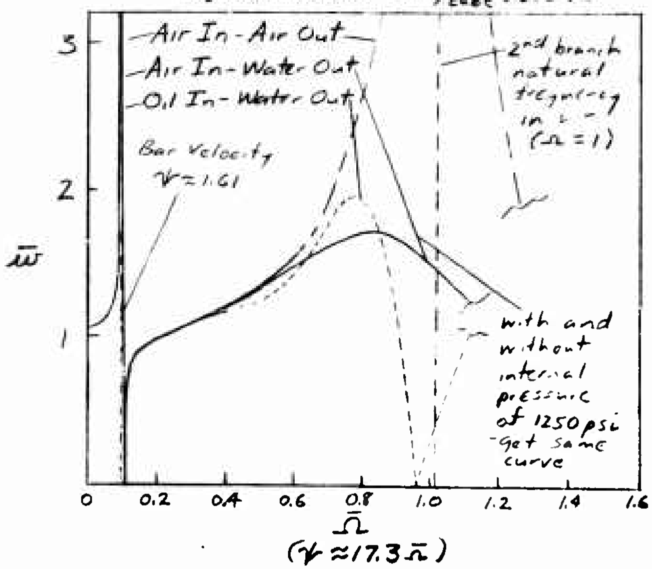


Fig. 7b Effect of Internal and External Fluid (ν=0.3, n=2, α=0.95, β=0.10) ρ_{air}/ρ_{tube}=0.00016 ρ_{oil}/ρ_{tube}=0.090

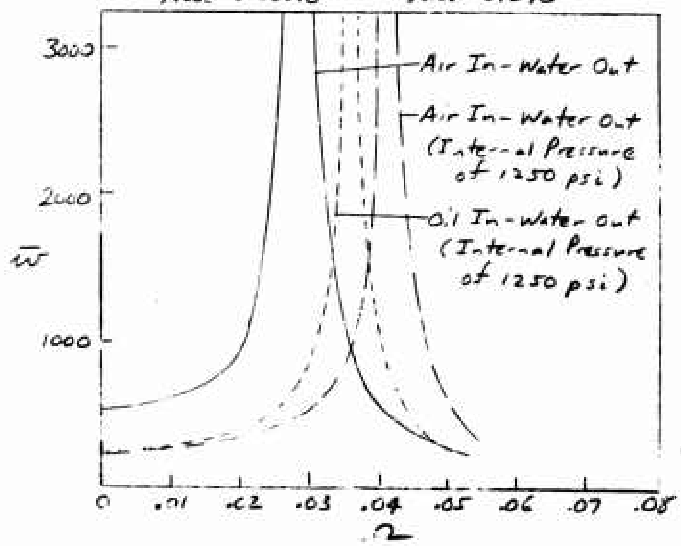
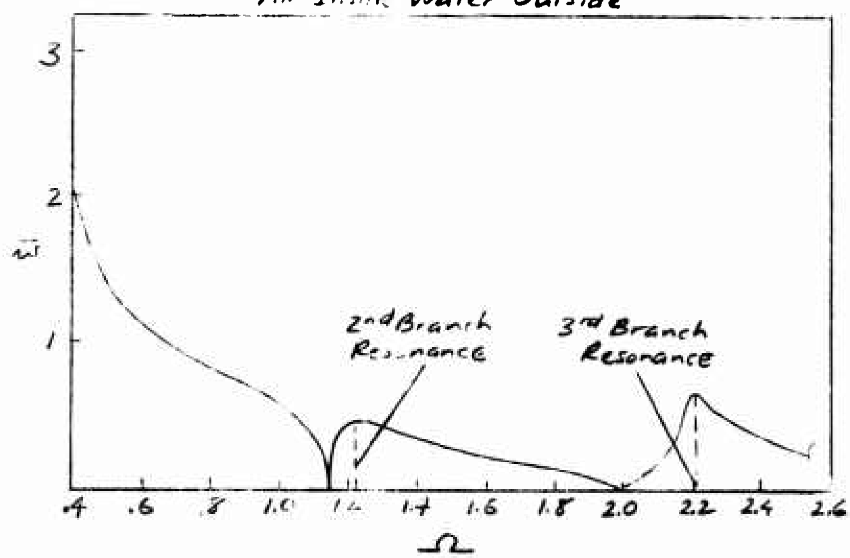


Fig. 7c Higher Branches (ν=0.3, n=2, α=0.95 β=0.10) Air Inside-Water Outside



$n=1,\ \beta=0.8$

Ω	R	P	θ_{mn}
0.10	1238	1.55	0
0.20	3734	23.14	0
0.40	558	13.70	1.02
0.50	300	8.77	1.07
0.80	93	4.00	1.04
1.20	4	0.26	1.02
1.30	148	10.00	1.02
1.50	50	3.90	1.01
1.60	20	1.61	1.01

$n=3,\ \beta=0.8$

Ω	R	P	θ_{mn}
0.40	0.021	0.0002	0.01
0.70	0.024	0.0012	0.71
1.40	0.041	0.0035	1.20
1.70	0.068	0.0067	1.13
2.00	0.222	0.0250	1.09
2.10	0.606	0.0700	1.08
2.20	0.314	0.0380	1.08
2.30	0.147	0.0183	1.07
2.50	0.036	0.0047	1.06

$n=5,\ \beta=0.8$

Ω	R	P	θ_{mn}
0.20	$.18\times10^{-5}$	26×10^{-8}	0
1.00	$.19\times10^{-5}$	12×10^{-6}	0.19
2.10	$.35\times10^{-5}$	48×10^{-6}	1.29
2.50	$.53\times10^{-5}$	$.80\times10^{-6}$	1.19
2.90	$.10\times10^{-4}$	$.17\times10^{-5}$	1.13
3.30	$.47\times10^{-4}$	$.86\times10^{-5}$	1.10
3.40	$.95\times10^{-4}$	$.18\times10^{-4}$	1.09
3.50	$.54\times10^{-4}$	$.10\times10^{-4}$	1.09
3.80	$.11\times10^{-4}$	$.23\times10^{-5}$	1.07

$n=1,\ \beta=3.0$

Ω	R	P	θ_{mn}
0.10	18.6	0.06	0
0.40	20.6	0.18	0
1.20	55.3	4.77	1.40
1.40	128.2	11.46	1.25
1.50	396.7	36.63	1.21
1.70	110.0	10.94	1.15
1.80	63.8	6.60	1.13
1.90	41.2	443	1.12
2.10	2.1	0.25	1.09

$n=3,\ \beta=3.0$

Ω	R	P	θ_{mn}
0.10	0.0083	0	0
0.60	0.0114	0.0002	0
1.10	0.0146	0.0018	1.37
1.60	0.0219	0.0025	1.39
1.90	0.0329	0.0040	1.24
2.20	0.0746	0.0097	1.17
2.40	0.3490	0.0483	1.14
2.50	0.2663	0.0380	1.13
3.10	0.0077	0.0013	1.08

$n=5,\ \beta=3.0$

Ω	R	P	θ_{mn}
0.10	$.15\times10^{-5}$	$.46\times10^{-7}$	0
2.00	$.21\times10^{-5}$	$.33\times10^{-6}$	1.54
2.30	$.26\times10^{-5}$	$.41\times10^{-6}$	1.36
2.90	$.52\times10^{-5}$	$.42\times10^{-6}$	1.19
3.20	$.97\times10^{-5}$	$.18\times10^{-5}$	1.15
3.40	$.20\times10^{-4}$	$.38\times10^{-5}$	1.13
3.60	$.83\times10^{-4}$	$.17\times10^{-4}$	1.12
3.80	$.21\times10^{-4}$	$.46\times10^{-5}$	1.10
4.0	$.10\times10^{-4}$	$.23\times10^{-5}$	1.09

$n=1,\ \beta=5.0$

Ω	R	P	θ_{mn}
0.10	2.6	0.0009	0
0.90	3.3	0.11	0
1.60	5.7	0.95	2.04
2.00	9.9	1.40	1.40
2.60	83.0	12.97	1.19
2.70	358.9	57.39	1.17
2.80	141.4	23.15	1.15
3.10	38.1	6.70	1.12
3.20	23.3	4.20	1.11

$n=3,\ \beta=5.0$

Ω	R	P	θ_{mn}
0.10	0.0028	0	0
0.90	0.0039	0.0001	0
1.50	0.0055	0.0012	0.72
2.70	0.0204	0.0035	1.24
3.10	0.0988	0.0181	1.17
3.20	0.3108	0.0582	1.16
3.30	0.1415	0.0270	1.14
3.40	0.0718	0.0140	1.13
3.90	0.0035	0.0008	1.10

Table 2. Exact Theory $\alpha=0.01$, $c_0/c_1=0.47$, $\rho_0/\rho=0.13$, $\nu=0.3$

$n=1, \beta=0.8$

Ω	R	P	θ_{mn}
0.10	34.3	.043	0
0.20	105.7	.655	0
0.30	39.7	.732	.69
0.40	14.6	.358	1.02
0.60	4.6	.155	1.06
0.90	.4	.021	1.04
1.00	1.9	.103	1.03
1.10	7.8	.447	1.03
1.20	2.3	.142	1.02
1.50	.2	.014	1.01

$n=3, \beta=0.8$

Ω	R	P	θ_{mn}
0.10	1.33	$.77\times10^{-3}$	0
0.30	1.40	$.80\times10^{-2}$	$.39\times10^{-3}$
0.90	2.45	.169	1.30
1.20	5.53	.437	1.26
1.30	10.48	.863	1.23
1.40	22.36	1.922	1.20
1.50	8.27	.742	1.17
1.60	4.48	.420	1.15
1.90	1.77	.189	1.10
2.30	.82	.103	1.07

$n=5, \beta=0.8$

Ω	R	P	θ_{mn}
0.10	.19	$.68\times10^{-4}$	0
1.10	.23	.019	.46
2.10	.45	.062	1.29
2.60	1.02	.157	1.17
2.80	2.00	.325	1.15
2.90	3.65	.609	1.13
3.00	8.51	1.453	1.12
3.10	4.20	.736	1.11
3.20	2.28	.409	1.11
3.70	.71	.142	1.08

$n=1, \beta=3.0$

Ω	R	P	θ_{mn}
0.10	1.36	$.69\times10^{-3}$	0
0.30	1.43	$.69\times10^{-2}$	0
0.50	1.60	.024	0
0.90	2.66	.330	2.30
1.10	4.20	.369	1.54
1.20	6.74	.582	1.40
1.30	17.69	1.542	1.31
1.40	12.29	1.099	1.25
1.60	2.85	.273	1.18
1.90	.48	.051	1.12

$n=3, \beta=3.0$

Ω	R	P	θ_{mn}
0.10	.60	$.24\times10^{-3}$	0
0.30	.61	$.23\times10^{-2}$	0
0.90	.81	.046	.024
1.30	1.21	.145	1.690
1.60	2.22	.254	1.390
1.80	5.92	.697	1.280
1.90	16.77	2.019	1.240
2.00	6.14	.759	1.210
2.10	3.13	.397	1.190
2.90	.19	.031	1.090

$n=5, \beta=3.0$

Ω	R	P	θ_{mn}
0.10	.13	$.41\times10^{-4}$	0
1.00	.16	$.63\times10^{-2}$	$.10\times10^{-2}$
2.00	.25	.040	1.54
2.70	.57	.095	1.23
3.10	1.81	.330	1.16
3.20	3.58	.668	1.15
3.30	7.34	1.400	1.14
3.40	2.97	.578	1.13
3.50	1.69	.337	1.12
4.00	.48	.106	1.09

$n=1, \beta=5.0$

Ω	R	P	θ_{mn}
0.10	.38	$.12\times10^{-3}$	0
0.30	.39	$.12\times10^{-2}$	0
1.30	.61	.077	0
1.80	1.02	.148	1.59
2.10	2.05	.293	1.34
2.30	6.59	.968	1.26
2.40	12.98	1.943	1.23
2.50	4.42	.675	1.21
2.60	2.49	.389	1.19
3.00	.64	.110	1.13

$n=3, \beta=5.0$

Ω	R	P	θ_{mn}
0.10	.22	$.65\times10^{-4}$	0
0.90	.26	$.73\times10^{-2}$	0
1.50	.36	.077	.72
2.10	.60	.099	1.52
2.40	1.04	.170	1.34
2.60	2.03	.339	1.27
2.70	3.78	.642	1.24
2.80	11.44	1.976	1.22
2.90	4.36	.768	1.20
3.80	.26	.055	1.10

$n=5, \beta=5.0$

Ω	R	P	θ_{mn}
0.10	.075	$.19\times10^{-4}$	0
0.90	.083	$.19\times10^{-2}$	0
2.10	.130	.030	2.03
2.60	.182	.036	1.48
3.20	.370	.075	1.26
3.50	.707	.151	1.20
3.70	1.570	.346	1.18
3.80	3.392	.760	1.16
3.90	5.170	1.178	1.15
4.00	2.140	.496	1.14

Table 3. Exact Theory $\alpha=0.50$, $c_o/c_r=0.47$, $\rho_o/\rho=0.13$, $\nu=0.3$

$n=1, \beta = 0.8$

Ω	R	P	θ_{mn}
0.10	35.95	.045	0
0.20	117.15	.726	0
0.30	38.02	.701	.69
0.40	14.81	.364	1.02
0.60	4.50	.152	1.06
0.80	.85	.037	1.04
0.90	1.76	.084	1.04
1.00	14.09	.743	1.03
1.10	3.03	.174	1.03
1.40	.14	.010	1.02

$n=3, \beta = 0.8$

Ω	R	P	θ_{mn}
0.10	4.74	$.27\times10^{-2}$	0
0.30	5.50	.031	$.39\times10^{-3}$
0.50	8.25	.168	.073
0.60	12.84	.438	.290
0.70	29.02	1.458	.714
0.80	27.91	1.744	1.118
0.90	13.00	.898	1.300
1.00	7.81	.570	1.329
1.20	3.91	.310	1.264
2.10	.63	.073	1.084

$n=5, \beta = 0.8$

Ω	R	P	θ_{mn}
0.10	.69	$.25\times10^{-3}$	0
0.50	.74	$.72\times10^{-2}$	$.72\times10^{-4}$
1.00	.96	.058	.191
1.60	1.91	.251	1.530
1.90	4.80	.644	1.372
2.00	8.76	1.192	1.328
2.10	8.56	1.185	1.290
2.20	4.50	.636	1.259
2.30	2.84	.410	1.232
3.50	.26	.050	1.088

$n=1, \beta = 3.0$

Ω	R	P	θ_{mn}
0.10	2.46	$.12\times10^{-2}$	0
0.40	2.83	.025	0
0.90	7.66	.949	2.30
1.00	10.40	.990	1.80
1.10	17.44	1.531	1.54
1.20	12.55	1.083	1.40
1.30	6.65	.580	1.31
1.40	4.19	.374	1.25
1.80	1.04	.107	1.13
1.90	.13	.014	1.12

$n=3, \beta = 3.0$

Ω	R	P	θ_{mn}
0.10	1.60	$.64\times10^{-3}$	0
0.50	1.87	.021	0
0.90	3.20	.182	.024
1.10	6.25	.769	1.372
1.20	8.39	1.049	1.740
1.30	12.54	1.498	1.689
1.40	10.81	1.251	1.572
1.50	6.08	.694	1.470
1.70	2.74	.317	1.330
2.70	.21	.032	1.105

$n=5, \beta = 3.0$

Ω	R	P	θ_{mn}
0.10	.47	$.14\times10^{-3}$	0
0.60	.51	$.61\times10^{-2}$	0
1.20	.67	.051	.078
1.80	1.17	.192	1.706
2.30	5.09	.812	1.359
2.40	8.95	1.442	1.319
2.50	5.19	.847	1.285
2.60	2.92	.484	1.257
3.20	.70	.131	1.152
3.90	.01	$.27\times10^{-2}$	1.096

$n=1, \beta = 5.0$

Ω	R	P	θ_{mn}
0.10	.81	$.26\times10^{-3}$	0
0.50	.87	$.75\times10^{-2}$	0
0.90	1.07	.036	0
1.30	1.72	.217	0
1.70	3.53	.539	1.75
1.80	5.47	.796	1.59
1.90	9.93	1.418	1.48
2.00	8.35	1.186	1.40
2.20	2.87	.416	1.30
3.00	.45	.077	1.13

$n=3, \beta = 5.0$

Ω	R	P	θ_{mn}
0.10	.56	$.16\times10^{-3}$	0
0.60	.61	$.69\times10^{-2}$	0
1.30	.91	.077	0
1.90	2.13	.371	1.77
2.10	4.62	.754	1.52
2.20	8.70	1.409	1.45
2.30	6.64	1.074	1.39
2.40	3.52	.574	1.34
3.10	.64	.118	1.17
3.60	.11	.023	1.12

$n=5, \beta = 5.0$

Ω	R	P	θ_{mn}
0.10	.27	$.66\times10^{-4}$	0
0.50	.28	$.17\times10^{-2}$	0
1.30	.34	.019	0
2.60	1.06	.208	1.48
2.90	2.74	.542	1.34
3.00	5.28	1.051	1.31
3.10	6.27	1.263	1.28
3.20	3.08	.627	1.26
3.30	1.89	.390	1.24
4.00	.47	.108	1.14

Table 4. Exact Theory $\alpha = 0.70$, $c_0/c_r = 0.47$, $\rho_0/\rho = 0.13$, $\nu = 0.3$

n	α	β	$\bar{\Omega}_{res}$	Ψ_{res}	$\frac{\bar{F}}{(\sigma_{max})^2}$	Branch
0	0.95	0.10	0.09	1.61	0.0008	1
		0.10	0.95	16.5	1.93	2
		0.50	0.46	1.59	0.07	1
		0.50	0.95	3.30	1.82	2
		3.00	0.60	0.35	0	1
		3.00	2.95	1.70	0.05	2
	0.90	0.10	0.09	1.61	0.0006	1
		0.10	0.98	17.4	2.28	2
		0.50	0.44	1.57	0.11	1
		0.50	1.00	3.56	2.24	2
		3.00	0.68	0.40	0	1
		3.00	2.85	1.69	0	2
1	0.95	0.10	0.004	0.08	0	1
		0.50	0.09	0.30	0	1
		3.00	0.57	0.33	0	1
	0.90	0.10	0.004	0.08	0	1
		0.50	0.09	0.30	0	1
		3.00	0.65	0.40	0	1
2	0.95	0.10	0.03	0.50	0.1×10^{-8}	1
		0.50	0.05	0.16	0	1
		3.00	0.50	0.28	0	1
	0.90	0.50	0.08	0.28	0	1
		3.00	0.57	0.34	0	1
4	0.95	0.10	0.17	2.95	2.3×10^{-8}	1
		0.50	0.17	0.60	0.07×10^{-8}	1
		3.00	0.38	0.22	0	1
	0.90	0.10	0.38	6.85	0.48×10^{-4}	1
		0.50	0.39	1.39	0.35×10^{-4}	1
		3.00	0.66	0.39	0	1

Table 5. Power Stress Ratios - Lower Branches
(Approximate Shell Theory)

n	α	β	$\bar{\Omega}_{res}$	ψ_{res}	$\bar{F}$	$\bar{\sigma}_x$	$\bar{\sigma}_\phi$	Branch
0	0.95	0.10	.95	16.5	1652	8.9	29.2	2
2	0.95	0.10	1.20	20.8	122	2.2	2.6	2
2	0.95	0.10	2.20	38.7	781	19.8	65.5	3
0	0.90	0.10	.98	17.4	1650	8.2	26.9	2
2	0.90	0.10	1.20	21.2	48	2.6	.7	2
2	0.90	0.10	2.20	39.7	760	20.0	66.4	3
1	0.90	0.10	0.60	10.8	77	7.7	2.9	2
1	0.90	0.10	1.40	25.0	800	12.4	40.9	3
0	0.95	0.50	0.95	3.3	1632	12.3	30.0	2
2	0.95	0.50	1.20	4.3	580	33.6	25.0	2
2	0.95	0.50	2.27	7.9	781	22.3	66.3	3
0	0.90	0.50	1.00	3.6	1630	11.0	27.0	2
2	0.90	0.50	1.27	4.5	75	5.9	3.8	2
2	0.90	0.50	2.27	8.1	778	22.6	67.3	3
1	0.90	0.50	0.72	2.6	778	38.4	15.4	2
1	0.90	0.50	1.44	5.1	796	15.6	41.6	3

Table 6. Power and Stress - Higher Branches
(Approximate Shell Theory)